AN EXTENSIVE ANALYSIS OF THE PARITY OF BROKEN 3–DIAMOND PARTITIONS

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ABSTRACT. In 2007, Andrews and Paule introduced the family of functions $\Delta_k(n)$ which enumerate the number of broken k–diamond partitions for a fixed positive integer k. Since then, numerous mathematicians have considered partitions congruences satisfied by $\Delta_k(n)$ for small values of k. In this work, we provide an extensive analysis of the parity of the function $\Delta_3(n)$, including a number of Ramanujan–like congruences modulo 2. This will be accomplished by **completely** characterizing the values of $\Delta_3(8n + r)$ modulo 2 for $r \in \{1, 2, 3, 4, 5, 7\}$ and any value of $n \geq 0$. In contrast, we conjecture that, for any integers $0 \leq B < A$, $\Delta_3(8(An+B))$ and $\Delta_3(8(An+B)+6)$ is infinitely often even and infinitely often odd. In this sense, we generalize Subbarao's Conjecture for this function Δ_3. To the best of our knowledge, this is the first generalization of Subbarao's Conjecture in the literature.

1. Introduction

Broken k-diamond partitions were introduced in 2007 by Andrews and Paule [2]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms. Namely,

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_k(n) q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})} \\ &= q^{(k+1)/12} \frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \quad k \geq 1,\end{aligned}$$

where we recall the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \quad (q = e^{2\pi i \tau}).$$

In their original work, Andrews and Paule proved that, for all $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod 3. \tag{1.1}$$

They also conjectured a few other congruences modulo 2 satisfied by certain families of broken k-diamond partitions.

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Since then, a number of authors have provided proofs of additional congruences satisfied by broken k-diamond partitions. Hirschhorn and Sellers [5] provided a new proof of (1.1) above as well as elementary proofs of the following parity results: For all $n \geq 0$,

$$\begin{aligned}
\Delta_1(4n+2) &\equiv 0 \pmod{2}, \\
\Delta_1(4n+3) &\equiv 0 \pmod{2}, \\
\Delta_2(10n+2) &\equiv 0 \pmod{2}, \quad \text{and} \\
\Delta_2(10n+6) &\equiv 0 \pmod{2}
\end{aligned}$$

The third result in the list above appeared in [2] as a conjecture while the other three did not. Soon after the publication of [5], Chan [3] provided a different proof of the parity results for Δ_2 mentioned above as well as a number of congruences modulo powers of 5. Subsequently, Paule and Radu [7] also proved a number of congruences modulo 5 for broken 2–diamond partitions, and they also shared conjectures related to broken 3–diamond partitions modulo 7 and broken 5–diamond partitions modulo 11. (Two of these conjectures have recently been proven by Xiong [12].)

Our goal in this work is to focus on parity results satisfied by $\Delta_3(n)$. The parity of this function has been studied, at least partially, by Radu and Sellers [10] who proved (among other things) that, for all $n \geq 0$,

$$\begin{aligned}
\Delta_3(14n+7) &\equiv 0 \pmod{2}, \\
\Delta_3(14n+9) &\equiv 0 \pmod{2}, \quad \text{and} \\
\Delta_3(14n+13) &\equiv 0 \pmod{2}.
\end{aligned} \tag{1.2}$$

We wish to greatly extend results such as those mentioned in (1.2). This will be accomplished by **completely** characterizing the values of $\Delta_3(8n+r)$ modulo 2 for $r \in \{1,2,3,4,5,7\}$ and any value of $n \geq 0$ by finding interesting relationships modulo 2 between the generating functions for $\Delta_3(8n+r)$ for these special values of r and classical q–series. We also note here that, while $\Delta_3(8n+r)$ is extremely "well–behaved" modulo 2 for the values $r \in \{1,2,3,4,5,7\}$, and satisfies numerous congruences modulo 2 in arithmetic progressions, we also believe that Δ_3 does **not** satisfy any Ramanujan–like congruences modulo 2 within any subprogression of $8n$ or $8n+6$. In this sense, we generalize Subbarao's Conjecture for this function Δ_3 by calling attention to the two arithmetic progressions $8n$ and $8n+6$. Our hope is that such an analysis will motivate others to complete similar work on other restricted parti! tion functions $f(n)$; namely, to locate a particular value A such that $f(An+r)$ has very nice parity properties for certain values of r while having **no** congruences modulo 2 within the other arithmetic progressions of the form $An+r$. (This seems to be a natural next step in the study of the parity of partition functions given the first author's recent proof of Subbarao's Conjecture [9].)

We note, in passing, that we also prove a number of parity results for $\Delta_3(4n+r)$ and $\Delta_3(2n+r)$ for various values of r. We begin with a characterization of the parity of $\Delta_3(2n+1)$ for any n.

Theorem 1.1.

$$\sum_{n=0}^{\infty}\Delta_3(2n+1)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)(1-q^{7n}) \pmod 2 \tag{1.3}$$

Remark 1.2. It should be noted that the coefficients of the power series representation of the product on the right–hand side of (1.3) can be completely classified modulo 2. First, we note that

$$q^{1/3}\prod_{n=1}^{\infty}(1-q^n)(1-q^{7n}) \equiv \sum_{m,n\in\mathbb{Z}} q^{\frac{(6m-1)^2+7(6n-1)^2}{24}} \pmod 2.$$

We then define

$$\sum_{n=0}^{\infty}a(n)q^n := \sum_{m,n\in\mathbb{Z}} q^{(6m-1)^2+7(6n-1)^2} \pmod 2.$$

Next note that $a(\nu)=0$ unless $\nu=24k+8$. If $\nu=24k+8=8(3k+1)$ we observe that

$$a(\nu)=\#\{(m,n)\in\mathbb{N}^2 : m^2+7n^2=8(3k+1), m,n\equiv 1 \pmod 2\}.$$

Moreover, if $7|\nu$, then $a(\nu)=a(\nu/7)$. This is clear because if $m^2+7n^2=7s$ then $7|m$ which implies that $(7\cdot(m/7))^2+7n^2=7s$ which implies that $n^2+7(m/7)^2=s$. Thus every solution to $m^2+7n^2=7s$ can be transformed into a solution of $m'^2+7n'^2=s$ where $m'=n$ and $n'=m/7$ and vice versa. Next, let n be a positive integer with $7\nmid n$ and let α be an integer greater than 2. Assume that there exists $x,y\in\mathbb{Z}$ with $x,y\equiv 1 \pmod 2$ such that

$$x^2+7y^2=2^\alpha n.$$

We note that the ring $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$ is a unique factorization domain. In particular, we have

$$2=\left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right).$$

Assume that

$$n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}\times(a_1+\sqrt{-7}b_1)^{\beta_1}\cdots(a_r+\sqrt{-7}b_r)^{\beta_r}\times(a_1-\sqrt{-7}b_1)^{\beta_1}\cdots(a_r-\sqrt{-7}b_r)^{\beta_r},$$

where $p_j, a_j\pm\sqrt{-7}b_j$ are primes. Set

$$2^\alpha=\Big(\frac{1+\sqrt{-7}}{2}\Big)^\alpha\Big(\frac{1-\sqrt{-7}}{2}\Big)^\alpha.$$

Note that

$$(x+\sqrt{-7}y)(x-\sqrt{-7}y)=2^\alpha n.$$

If r_j is maximal such that $p_j^{r_j}|(x+\sqrt{-7}y)$, then $p_j^{r_j}|(x-\sqrt{-7}y)$ which implies that $2r_j=\alpha_j$ for $j=1,\dots,s$. It follows that

$$(a_i+\sqrt{-7}b_i)^{j_i}(a_i-\sqrt{-7}b_i)^{\beta_i-j_i}|(x+\sqrt{-7}y),$$

for some $j_i = 0, \cdots, \beta_i$ and $i = 1, \ldots, r$. Furthermore, either

$$\Big(\frac{1+\sqrt{-7}}{2}\Big)\Big(\frac{1-\sqrt{-7}}{2}\Big)^{\alpha-1}$$

or

$$\Big(\frac{1-\sqrt{-7}}{2}\Big)\Big(\frac{1+\sqrt{-7}}{2}\Big)^{\alpha-1}$$

divides $x + \sqrt{-7}y$. These are the only possibilites that guarantee that x and y are odd. Consequently, in total we have $2\prod_{j=1}^{r}(1+\beta_j)$ possibilities for $x+\sqrt{-7}y$. If out of this we choose only those with $x \geq 0$ we obtain $\prod_{j=1}^{r}(1+\beta_j)$ possibilities. This implies that

$$a(2^\alpha n) = \prod_{j=1}^{r}(1+\beta_j)$$

where $7 \nmid n$ and

$$a(2^\alpha 7^k n) = \prod_{j=1}^{r}(1+\beta_j).$$

Thus,

$$a(2^\alpha 7^k n) \equiv 1 \pmod 2$$

iff β_j is even for all j or equivalently if n is a square. Next note that $2^\alpha 7^k t^2 \equiv 8 \pmod{24}$ iff $3 \nmid t$ and $\alpha - 3$ is even and nonnegative. This implies that

$$\sum_{n=0}^{\infty} a(n)q^n \equiv \sum_{n=0}^{\infty} A(n)q^n \pmod 2,$$

where

$$A(n) := \begin{cases} 1 & \text{if } n = 2^3t^2 \text{ or } n = 2^3 7 t^2,\ 3 \nmid t, k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$q^{1/3}\prod_{n=1}^{\infty}(1-q^n)(1-q^{7n}) \equiv \sum_{n=0}^{\infty} A(n)q^{n/24} = \sum_{k=0}^{\infty} A(24k+8)q^{k+1/3}.$$

Hence,

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^{7n}) \equiv \sum_{k=0}^{\infty} A(24k+8)q^k = \sum_{t\geq 0, 3\nmid t} q^{\frac{7t^2-1}{3}} + q^{\frac{t^2-1}{3}}.$$

Thanks to the above analysis, we have the following:

Corollary 1.3. *For all* $n \geq 0$, $\Delta_3(2n+1) \equiv 1 \pmod 2$ *if and only if* $3n+1 = t^2$ *or* $3n+1 = 7t^2$.

Notice that the three congruences mentioned in (1.2) follow almost immediately from this characterization given in Corollary 1.3. For example, the above work implies that we need to consider whether $3(7n+3)+1$ or $21n+10$ can be represented as t^2 or $7t^2$ for some integer t in order to determine the parity of $\Delta_3(14n+7)$. Note that $21n+10$ is not divisible by 7, so it cannot be written in the form $7t^2$. Moreover, $21n+10$ can never be square because $21n + 10 \equiv 3 \pmod 7$ and 3 is a quadratic nonresidue modulo 7. In analogous fashion, $\Delta_3(14n+9) \equiv 0 \pmod 2$ because 6 is a quadratic nonresidue modulo 7, and $\Delta_3(14n+13) \equiv 0 \pmod 2$ because 5 is a quadratic nonresidue modulo 7.

We now consider parity results satisfied by $\Delta_3(4n+r)$ for various values of r.

Theorem 1.4.

$$\sum_{n=0}^{\infty} \Delta_3(4n)q^n \equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{7n})} \pmod 2$$

Theorem 1.5.

$$\sum_{n=0}^{\infty} \Delta_3(4n+2)q^n \equiv q\prod_{n=1}^{\infty} \frac{(1-q^{7n})^5}{(1-q^{n})} \pmod 2$$

Theorem 1.6.

$$\sum_{n=0}^{\infty} \Delta_3(4n+3)q^n \equiv \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{14n}) \pmod 2$$

Remark 1.7. A few remarks are in order regarding Theorem 1.6. First, note that the product on the right–hand side of the congruence is an even function of q. This implies that, for all $n \geq 0$, $\Delta_3(4(2n+1)+3) \equiv \pmod 2$ or $\Delta_3(8n+7) \equiv 0 \pmod 2$. Secondly, note that the right–hand side of Theorem 1.6 is the same as the right–hand side in Theorem 1.1 except with q replaced by q^2. Therefore, we can completely characterize the values of $\Delta_3(4n+3)$ modulo 2 via the remarks made regarding Theorem 1.1.

Our last set of theorems provides information about the parity of $\Delta_3(8n+r)$ for a number of values of r.

Theorem 1.8.

$$\sum_{n=0}^{\infty} \Delta_3(8n+1)q^n \equiv \prod_{n=1}^{\infty} (1-q^{2n}) \pmod 2$$

Remark 1.9. As with Theorem 1.6, it is clear that the right–hand side in Theorem 1.8 is an even function of q. Thus, we know that, for all $n \geq 0$, $\Delta_3(16n+9) \equiv 0 \pmod 2$ immediately. But we actually can say more. Thanks to Euler's Pentagonal Number Theorem [1, Corollary 1.7], we know

$$\prod_{n=1}^{\infty} (1-q^{2n}) = \sum_{m\in\mathbb{Z}}^{\infty} (-1)^m q^{m(3m-1)}.$$

Therefore, we can explicitly state when $\Delta_3(8n+1)$ is even or odd; namely, for any $n \geq 0$, $\Delta_3(8n+1)$ is odd if and only if $n = m(3m-1)$ for some integer m. This is equivalent to saying $\Delta_3(8n+1)$ is odd if and only if $12n+1$ is a perfect square. This means we can write down numerous Ramanujan–like congruences modulo 2 within the arithmetic progression $8n+1$ with ease.

Theorem 1.10.

$$\sum_{n=0}^{\infty}\Delta_3(8n+2)q^n \equiv q\prod_{n=1}^{\infty}(1-q^n)(1-q^{28n}) \pmod 2$$

Theorem 1.11.

$$\sum_{n=0}^{\infty}\Delta_3(8n+3)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)(1-q^{7n}) \pmod 2$$

Remark 1.12. Given Theorem 1.1, we see that Theorem 1.11 clearly implies that, for all $n \geq 0$, $\Delta_3(8n+3) \equiv \Delta_3(2n+1) \pmod 2$, an attractive "internal" congruence satisfied by Δ_3. We will briefly mention this congruence again in our concluding remarks below.

Theorem 1.13.

$$\sum_{n=0}^{\infty}\Delta_3(8n+4)q^n \equiv \prod_{n=1}^{\infty}(1-q^{4n})(1-q^{7n}) \pmod 2$$

A remark is in order regarding Theorems 1.10 and 1.13. We have

$$q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n-1)^2/24} \equiv \sum_{n=-\infty}^{\infty} q^{(6n-1)^2/24} \pmod 2.$$

Consequently,

$$q^{\frac{11}{24}}\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{7n}) \equiv \sum_{n,m\in\mathbb{Z}} q^{\frac{4(6n-1)^2+7(6m-1)^2}{24}} \pmod 2$$

and

$$q^{\frac{29}{24}}\prod_{n=1}^{\infty}(1-q^{n})(1-q^{28n}) \equiv \sum_{n,m\in\mathbb{Z}} q^{\frac{(6n-1)^2+28(6m-1)^2}{24}} \pmod 2.$$

Next we note that

$$n^2+7m^2 \equiv 11 \pmod{24} \Leftrightarrow n=2k, k,m \equiv \pm 1 \pmod 6$$

and

$$n^2+7m^2 \equiv 29 \pmod{24} \Leftrightarrow m=2k, n,k \equiv \pm 1 \pmod 6.$$

For given x with $x \equiv 11 \pmod{24}$ the set of solutions (n,m) such that $4n^2+7m^2=x$ can be partitioned into equivalence classes and two solutions (n_1,m_1) and (n_2,m_2) are equivalent iff $n_1=\pm n_2$ and $m_1=\pm m_2$. In particular each equivalence class has exactly 4 elements and there is only one solution (n_1,m_1) in each class such that $n_1=2k_1$ and $(k_1,m_1)\equiv(-1,-1) \pmod 6$. This implies in particular that for

$$\sum_{n=0}^{\infty} b(n)q^n := \sum_{n,m\in\mathbb{Z}} q^{n^2+7m^2} \tag{1.4}$$

we have

$$\frac{1}{4}\sum_{n=0}^{\infty} b(24n+11)q^{24n+11} = \sum_{n,m\in\mathbb{Z}} q^{4(6n-1)^2+7(6m-1)^2}.$$

This implies that

$$\frac{1}{4}q^{\frac{11}{24}}\sum_{n=0}^{\infty} b(24n+11)q^n = \sum_{n,m\in\mathbb{Z}} q^{\frac{4(6n-1)^2+7(6m-1)^2}{24}}.$$

In a similar fashion we conclude that

$$\frac{1}{4}q^{\frac{29}{24}}\sum_{n=0}^{\infty} b(24n+29)q^n = \sum_{n,m\in\mathbb{Z}} q^{\frac{(6n-1)^2+28(6m-1)^2}{24}}.$$

Because of these two relations we observe that in order to understand $\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{7n})$ and $\prod_{n=1}^{\infty}(1-q^{n})(1-q^{28n})$ modulo 2 we need to understand $b(n)$ in (1.4) for n odd. By [4, p. 61, Lemma 3.25] we know that, for $m>1$ and odd with $7\nmid m$,

$$|\{x,y\in\mathbb{Z}:\gcd(x,y)=1, x^2+7y^2=m\}| = 2\prod_{p|m}\left(1+\left(\frac{-7}{p}\right)\right).$$

Let $m=m_s^2m_f$ with m_f squarefree. Then we observe immediately that

$$|\{x,y\in\mathbb{Z}: x^2+7y^2=m\}| = 2\sum_{d|m_s}\prod_{p|\frac{m}{d^2}}\left(1+\left(\frac{-7}{p}\right)\right).$$

Consequently,

$$b(m) = 2\sum_{d|m_s}\prod_{p|\frac{m}{d^2}}\left(1+\left(\frac{-7}{p}\right)\right). \tag{1.5}$$

By using the fact that

$$|\{x,y\in\mathbb{Z}: x^2+7y^2=7^{\alpha}n\}| = |\{x,y\in\mathbb{Z}: x^2+7y^2=n\}|$$

one can lift the restriction that $7\nmid m$. From (1.5) we observe that $\frac{b(m)}{2}$ is multiplicative for odd m. Because of (1.5), we know for prime $p\geq 3$ that

$$\begin{aligned} b(p^{2\alpha+1}) &= 2(\alpha+1)\left(1+\left(\frac{-7}{p}\right)\right)\\ b(p^{2\alpha}) &= 2\left(\alpha\left(1+\left(\frac{-7}{p}\right)\right)+1\right). \end{aligned}$$

This now leads to two corollaries which give a characterization of the values of $\Delta_3(8n+2)$ and $\Delta_3(8n+2)$, modulo 2, in terms of this function $b(n)$ just described:

Corollary 1.14. *For all $n\geq 0$, $\Delta_3(8n+2)\equiv \frac{1}{4}b(24n+29) \pmod 2$.*

Corollary 1.15. *For all $n\geq 0$, $\Delta_3(8n+4)\equiv \frac{1}{4}b(24n+11) \pmod 2$.*

Theorem 1.16.

$$\sum_{n=0}^{\infty}\Delta_3(8n+5)q^n \equiv \prod_{n=1}^{\infty}(1-q^{14n}) \pmod 2$$

Remark 1.17. As was discussed after Theorem 1.8, we can employ Euler's Pentagonal Number Theorem here as well to obtain a similar classification result. We can also easily see that, for all $n \geq 0$, $\Delta_3(16n+13) \equiv 0 \pmod 2$ since the right–hand side of Theorem 1.16 is an even function of q. In similar fashion, since the right–hand side is also a function of q^7, we can say that, for all $n \geq 0$, $\Delta_3(56n+r) \equiv 0 \pmod 2$ for $r \in \{13, 21, 29, 37, 45, 53\}$.

2. Proof of the Congruences

Let

$$f = \sum a(n)q^n := \frac{\eta(6z)\eta(21z)}{\eta^3(3z)\eta(42z)}$$

and

$$\phi := \eta(8z)^{72}.$$

Then

$$a(n) = \Delta_3\Big(\frac{n+1}{3}\Big).$$

Let

$$g = \sum b(n)q^n.$$

For χ a character we define

$$g_\chi := \sum \chi(n)b(n)q^n$$

and for $D \in \mathbb{Z}$, $\epsilon(n) := \left(\frac{D}{n}\right)$ let

$$g_D := g_\epsilon.$$

Define the U_d-operator by

$$U_d g := \sum b(dn)q^n.$$

We need that for $F := \sum A(n)q^n$, $G := \sum B(n)q^{nN}$ and χ a character modulo N we have

$$(FG)_\chi(z) = F_\chi(z)G(z) \tag{2.1}$$

and

$$U_N(FG) = G(z/N)(U_N F)(z). \tag{2.2}$$

One verifies that our congruences are equivalent to the following:

$$\begin{array}{cc}
\text{Thm. 1.1:} & U_2 f \equiv \eta(3z)\eta(21z) \pmod 2 \\
\text{Thm. 1.6:} & U_4 f \equiv \eta(6z)\eta(42z) \pmod 2 \\
\text{Thm. 1.11:} & U_8 f \equiv \eta(3z)\eta(21z) \pmod 2 \\
\text{Thm. 1.5:} & \frac{1}{2}(f_4 + f_{-4}) \equiv \frac{\eta^5(84z)}{\eta(12z)} \pmod 2 \\
\text{Thm. 1.4:} & \frac{1}{2}(f_4 - f_{-4}) \equiv \frac{\eta^5(12z)}{\eta(84z)} \pmod 2 \\
\text{Thm. 1.8:} & \frac{1}{2}([U_2 f]_4 + [U_2 f]_{-4}) \equiv \eta(24z) \pmod 2 \\
\text{Thm. 1.16:} & \frac{1}{2}([U_2 f]_4 - [U_2 f]_{-4}) \equiv \eta(168z) \pmod 2 \\
\text{Thm. 1.10:} & \frac{1}{4}(f_4 + f_{-4} - f_8 - f_{-8}) \equiv \eta(24z)\eta(672z) \pmod 2 \\
\text{Thm. 1.13:} & \frac{1}{4}(f_4 - f_{-4} - f_8 + f_{-8}) \equiv \eta(96z)\eta(168z) \pmod 2.
\end{array} \tag{2.3}$$

Next note that ϕ is a series in powers of q^8. Let $\phi^{(s)}(z) := \phi(z/s)$. In particular note that $\phi^{(s)}(z)$ is a series in powers of $q^{8/s}$. Using (2.1) and (2.2) we find

(2.4)
$$\begin{gathered}
U_2(f\phi) = \phi^{(2)}U_2 f\\
U_4(f\phi) \equiv \phi^{(4)}U_4 f\\
U_8(f\phi) \equiv \phi^{(8)}U_8 f\\
\tfrac{1}{2}((f\phi)_4 + (f\phi)_{-4}) = \phi\cdot\tfrac{1}{2}(f_4 + f_{-4})\\
\tfrac{1}{2}((f\phi)_4 - (f\phi)_{-4}) = \phi\cdot\tfrac{1}{2}(f_4 - f_{-4})\\
\tfrac{1}{2}([U_2(f\phi)]_4 + [U_2(f\phi)]_{-4}) = \tfrac{1}{2}([\phi^{(2)}U_2 f]_4 + [\phi^{(2)}U_2 f]_{-4}) = \phi^{(2)}\cdot\tfrac{1}{2}([U_2 f]_4 + [U_2 f]_{-4})\\
\tfrac{1}{2}([U_2(f\phi)]_4 - [U_2(f\phi)]_{-4}) = \tfrac{1}{2}([\phi^{(2)}U_2 f]_4 - [\phi^{(2)}U_2 f]_{-4}) = \phi^{(2)}\cdot\tfrac{1}{2}([U_2 f]_4 - [U_2 f]_{-4})\\
\tfrac{1}{4}((f\phi)_4 + (f\phi)_{-4} - (f\phi)_8 - (f\phi)_{-8}) = \phi\cdot\tfrac{1}{4}(f_4 + f_{-4} - f_8 - f_{-8})\\
\tfrac{1}{4}((f\phi)_4 - (f\phi)_{-4} - (f\phi)_8 + (f\phi)_{-8}) = \phi\cdot\tfrac{1}{4}(f_4 - f_{-4} - f_8 + f_{-8})
\end{gathered}$$

Recall that $\frac{\eta(\tau)^2}{\eta(2\tau)} \equiv 1 \pmod 2$. Then because of (2.4), (2.3) is equivalent to:

(2.5)
$$\begin{array}{ll}
\text{Thm. 1.1:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot U_2(f\phi) \equiv \phi^{(2)}\eta(3z)\eta(21z) \pmod 2\\
\text{Thm. 1.6:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot U_4(f\phi) \equiv \phi^{(4)}\eta(6z)\eta(42z) \pmod 2\\
\text{Thm. 1.11:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot U_8(f\phi) \equiv \phi^{(8)}\eta(3z)\eta(21z) \pmod 2\\
\text{Thm. 1.5:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{2}((f\phi)_4 + (f\phi)_{-4}) \equiv \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^{-2}\cdot\phi\cdot\frac{\eta^5(84z)}{\eta(12z)} \pmod 2\\
\text{Thm. 1.4:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{2}((f\phi)_4 - (f\phi)_{-4}) \equiv \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^{-2}\cdot\phi\cdot\frac{\eta^5(12z)}{\eta(84z)} \pmod 2\\
\text{Thm. 1.8:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{2}([U_2(f\phi)]_4 + [U_2(f\phi)]_{-4}) \equiv \phi^{(2)}\cdot\eta(24z) \equiv \phi^{(2)}\cdot\eta(12z)^2 \pmod 2\\
\text{Thm. 1.16:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{2}([U_2(f\phi)]_4 - [U_2(f\phi)]_{-4}) \equiv \phi^{(2)}\cdot\eta(168z) \equiv \phi^{(2)}\cdot\eta(84z)^2 \pmod 2\\
\text{Thm. 1.10:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{4}((f\phi)_4 + (f\phi)_{-4} - (f\phi)_8 - (f\phi)_{-8}) \equiv \phi\cdot\eta(24z)\eta(672z) \pmod 2\\
\text{Thm. 1.13:} & \left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4\cdot \tfrac{1}{4}((f\phi)_4 - (f\phi)_{-4} - (f\phi)_8 + (f\phi)_{-8}) \equiv \phi\cdot\eta(96z)\eta(168z) \pmod 2.
\end{array}$$

Denote by $M_k(N,\chi)$ the set of weak modular forms of weight k and character χ for the group $\Gamma_0(N)$. By [6, Th. 1.64] we have that $f\phi \in M_{35}\Big(504, \big(\frac{-1}{d}\big)\Big)$ and $\left(\frac{\eta(\tau)^2}{\eta(2\tau)}\right)^4 \in M_2(4,\mathrm{id})$. Furthermore, by [6, Prop. 2.8] we have that if $g \in M_k(N,\chi)$ and $\left(\frac{D}{n}\right)$ is a character modulo m, then $g_D \in M_k(Nm^2,\chi)$. By [6, Prop. 2.22], if $g \in M_k(N,\chi)$ and $d|N$, then $U_d f \in M_k(N,\chi)$. This implies that the left hand side of the relations in (2.5) in the first three lines are in $M_{37}\Big(504, \big(\frac{-1}{d}\big)\Big)$, in the next four lines they are in $M_{37}\Big(504\cdot 4^2, \big(\frac{-1}{d}\big)\Big)$ and in the last two lines they are in $M_{37}\Big(504\cdot 8^2, \big(\frac{-1}{d}\big)\Big)$. One can check the same holds for the functions on the right hand side using [6, Th. 1.64]. Using a generalization of Sturm's theorem [11], namely [6, Th. 2.58], we find that the first three identities hold if they hold for the first $\frac{35}{12}\times 1152 \sim 3360$ coefficients in their q-expansion. Similarly, the next four identities hold if they hold for $\frac{35}{12}\times 18432 \sim 53760$ coefficients in their q-expansions. Finally for the last two identities on needs to check about 215040 coefficients modulo 2.

Remark 2.1. An alternative method to prove these identities in their original form is using the approach from [8] which leads to a less elegant proof but more direct on the problem. We calculated using this method that we do not need to compute more that 1056 coefficients for any of the identities.

3. Closing Comments

We close this note by sharing a conjectured infinite family of "internal" congruences satisfied by $\Delta_3(n)$ modulo powers of 2:

Conjecture: Let

$$\lambda_\alpha = \begin{cases} \frac{2^{\alpha+1}+1}{3} & \text{if } \alpha \text{ is even,} \\ \frac{2^{\alpha}+1}{3} & \text{if } \alpha \text{ is odd.} \end{cases}$$

Then, for all $\alpha \geq 1$ and $n \geq 0$,

$$\Delta_3(\lambda_\alpha)\Delta_3(2^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \Delta_3(\lambda_{\alpha+2})\Delta_3(2^{\alpha}n + \lambda_\alpha) \pmod{2^\alpha}$$

and

$$\Delta_3(\lambda_\alpha) \equiv 1 \pmod{2}.$$

The case $\alpha = 1$ of this conjecture was proven above; namely, in Remark 1.12, we noted that

$$\Delta_3(8n+3) \equiv \Delta_3(2n+1) \pmod{2}$$

for all $n \geq 0$.

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