Symbolic solutions of first-order algebraic ODEs

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Algebraic ordinary differential equations are described by polynomial relations between the unknown function and its derivatives. There are no general solution methods available for such differential equations. However, if the hypersurface determined by the defining polynomial of an algebraic ordinary differential equation admits a parametrization, then solutions can be computed and the solvability in certain function classes may be decided. After an overview of methods developed in the last decade we present a new and rather general method for solving algebraic ordinary differential equations.

1 Introduction

Consider the field of rational functions $\mathbb{K}(x)$ for a field $\mathbb{K}$. Let $'$ be the uniquely defined derivation on $\mathbb{K}(x)$ with constant field $\mathbb{K}$ and $x' = 1$. Then $\mathbb{K}(x)$ is a differential field. By $\mathbb{K}(x)\{y\}$, we denote the ring of differential polynomials in y over $\mathbb{K}(x)$. Its elements are polynomials in y and the derivatives of y, i. e. $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \ldots]$. An algebraic ordinary differential equation (AODE) is of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0\,,$$

where $F \in \mathbb{K}(x)\{y\}$ and F is also polynomial in x. The AODE is called autonomous if $F \in \mathbb{K}\{y\}$, i. e. if the coefficients of F do not depend on the variable of differentiation x. For a given AODE we are interested in deciding whether it has rational or radical solutions and, in the affirmative case, determining all of them.

In order to define the notion of a general solution, we go a little more into detail. Let Σ be a prime differential ideal in $\mathbb{K}(x)\{y\}$. Then we call η a generic zero of Σ if for any

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differential polynomial P, we have $P(\eta) = 0 \Leftrightarrow P \in \Sigma$. Such an η exists in a suitable extension field.

Let F be an irreducible differential polynomial of order n. Then $\{F\}$, the radical differential ideal generated by F, can be decomposed into two parts. There is one component where the separant $\frac{\partial F}{\partial y^{(n)}}$ also vanishes. This component represents the singular solutions. The component we are interested in is the one where the separant does not vanish. It is a prime differential ideal $\Sigma_F := \{F\} : \langle \frac{\partial F}{\partial y^{(n)}} \rangle$ and represents the general component (see for instance Ritt [15]) comprising the regular solutions. A generic zero of Σ_F is called a general solution of $F = 0$. We call it a rational general solution if it is of the form $y = \frac{a_k x^k + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$, where the a_i and b_i are algebraic over $\mathbb{K}$.

Here we consider only first order AODEs. For solving differential equations $G(x, y, y') = 0$ or $F(y, y') = 0$, we will look at the corresponding surface $G(x, y, z) = 0$ or curve $F(y, z) = 0$, respectively, where we replace y' by a transcendental variable z.

A plane algebraic curve $\mathcal{C} = \{(a, b) \in \mathbb{K}^2 \mid f(a, b) = 0\}$ over $\mathbb{K}$ is a one-dimensional algebraic variety, i. e. the zero set of a square-free bivariate polynomial $f \in \mathbb{K}[x, y]$. The polynomial f is called the defining polynomial of $\mathcal{C}$. An important aspect of algebraic curves is their parametrizability. Consider an irreducible plane algebraic curve defined by an irreducible polynomial f. A tuple of rational functions $\mathcal{P}(t) = (r(t), s(t))$ is called a rational parametrization of the curve if $f(r(t), s(t)) = 0$ and not both $r(t)$ and $s(t)$ are constant. A rational parametrization can be considered as a rational map $\mathcal{P}(t) : \mathbb{A} \to \mathcal{C}$. By abuse of notation we also call this map a (rational) parametrization. Later we will see other kinds of parametrizations. We call a parametrization $\mathcal{P}(t)$ proper if it is a birational map or, in other words, if for almost every point (x, y) on the curve we find exactly one t such that $\mathcal{P}(t) = (x, y)$. Parametrizations of higher dimensional algebraic varieties are defined in a similar way.

In this paper we use parametrizations of curves for solving AODEs. Hubert [8] already studies solutions of AODEs of the form $F(x, y, y') = 0$. She gives a method for finding a basis of the general solution of the equation by computing a Gröbner basis of the prime differential ideal of the general component. The solutions, however, are given implicitly. Later Feng and Gao [2, 3] start using parametrizations for solving first order autonomous AODEs, i. e., AODEs of the form $F(y, y') = 0$. They provide an algorithm for actually solving such AODEs with coefficients in $\mathbb{Q}$ by using rational parametrizations of the algebraic curve $F(x, y) = 0$. The key fact, they are proving, is that any rational solution of the AODE gives a proper parametrization of the corresponding algebraic curve. For this they use a degree bound derived in Sendra and Winkler [19]. On the other hand, if a proper parametrization of the algebraic curve fulfills certain requirements, Feng and Gao can generate a rational solution of the AODE. All proper parametrizations of a plane algebraic curve are related by linear transformations. So one only needs to check whether the given parametrization can be linearly transformed into a parametrization whose second component is the derivative of the first. From the rational solution it is then possible to create a rational general solution by shifting the variable by a constant. This approach takes advantage of the well known theory of algebraic curves and rational parametrizations (see for instance [20, 21]). For this reason, we speak of the algebro-geometric solution method for AODEs.

In [1] Aroca, Cano, Feng and Gao give a necessary and sufficient condition for an autonomous AODE to have an algebraic solution. They also provide a polynomial time algorithm to find such a solution if it exists. Their solution, however, is implicit, whereas we are interested in explicit solutions.

In Section 2 we give a brief overview of the algebro-geometric solution method for AODEs. In Section 3 we restrict to the case of first-order autonomous AODEs, but try to extend the results to radical solutions using radical parametrizations (see Section 3.2). We do so by investigating a procedure based on a rather general form of parametrizations. In case the parametrization is rational, this general procedure simply contains the solution method for rational solutions as a special case (see Section 3.1). As shown in [1, 2] it is enough to look for a single non-trivial solution, for if $y(x)$ is a solution, so is $y(x+c)$ for a constant c and the latter is also a general solution. In Section 3.3 we give examples of non-radical solutions that can be found by the given procedure. Finally in Section 3.4 we look for advantages of the procedure and compare it to existing algorithms.

2 Overview of the algebro-geometric solution method for AODEs

Consider an autonomous first-order AODE $F(y, y') = 0$. Suppose this AODE has a rational regular solution $y(x)$. Then $\mathcal{P}(t) = (y(t), y'(t))$ is a rational parametrization of the curve $\mathcal{P}$ definded by $F(y, z) = 0$ in the affine plane over the field $\mathbb{K}$. Indeed, $\mathcal{P}$ is a proper parametrization, i. e., a birational map from the affine line onto $\mathcal{C}$. So if $\mathcal{C}$ does not admit a rational parametrization, then the given AODE has no rational solution. But if $\mathcal{C}$ is rationally parametrizabel, then there are infinitely many proper parametrizations, all of which can be determined from a single one by linear transformations. So we can set up a general proper parametrization and determine the existence of a solution of the AODE by solving a system of algebraic equations in the unknown coefficients of the linear transformation. This was first realized by Feng and Gao in [2] and later elaborated in [3]. The decision algorithm is based on exact degree bounds for parametrizations, as derived in [19]. For a detailled discussion of parametrizations, we refer to [20].

In [11] and [13] Ngô and Winkler have generalized this solution method for autonomous first-order AODEs to general first-order AODEs. For an overview of this algebro-geometric method, we refer to [12].

Example 2.1.
Consider the non-autonomous first-order AODE

$$F(x, y, y') = y'^2 + 3y' - 2y - 3x = 0.$$

The rational general solution of $F(x,y,y')=0$ is $y=\frac{1}{2}((x+c)^2+3c)$, where c is an arbitrary constant. The separant of F is $S=2y'+3$. So the singular solution of F is $y=-\frac{3}{2}x-\frac{9}{8}$. But how do we get this rational general solution? Let us repeat briefly the computation described in [12].

The solution surface $z^2+3z-2y-3x=0$ has the proper rational parametrization

$$\mathcal{P}(s,t) \;=\; \left(\frac{st+2s+t^2}{s^2},\; -\frac{3s+t^2}{s^2},\; \frac{t}{s}\right).$$

The original differential condition $F=0$ can be transformed into the so-called associated system, which is a system of differential conditions on the parameters:

$$s'=st, \qquad t'=s+t^2.$$

The rational solutions of the original AODE are in 1-1 correspondence to the rational solutions of the associated system. Observe that these conditions are autonomous, also first-order, and of degree 1 in the derivatives of the parameters. All this will be true in general. Now we can consider the irreducible invariant algebraic curves of the associated system:

$$G(s,t)=s, \qquad G(s,t)=t^2+2s, \qquad G(s,t)=s^2+ct^2+2cs.$$

Invariant algebraic curves are candidates for generating rational solutions of the associated system. The third algebraic curve $s^2+ct^2+2cs=0$ depends on a transcendental parameter c. It can be parametrized by

$$\mathcal{Q}(x)=\left(-\frac{2c}{1+cx^2},-\frac{2cx}{1+cx^2}\right).$$

The rational solution of the associated system is

$$s(x)=-\frac{2c}{1+cx^2}, \qquad t(x)=-\frac{2cx}{1+cx^2}.$$

In general we might have to apply a linear transformation in order to get a solution of the differential problem. Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. This rational general solution of the associated system can be tranformed into a rational general solution of $F(x,y,y')=0$; in this case

$$y=\frac{1}{2}x^2+\frac{1}{c}x+\frac{1}{2c^2}+\frac{3}{2c},$$

which, after a change of parameter, can be written as $y=\frac{1}{2}((x+c)^2+3c)$.

Of course, not all AODEs are first-order. In [6, 7] Huang, Ngô and Winkler have described how some of these methods can be generalized to AODEs of higher order. Also, one might try to find transformations of the ambient space which transform non-autonomous AODEs into autonomous ones. Indeed, the AODE considered in Example

2.1 can be transformed into the autonomous equation $y'^2 - 2y - \frac{9}{4} = 0$ in such a way that the rational solutions are in 1-1 correspondence. A complete characterization of affine transformations preserving rational solvability has been achieved in [10].

In the present paper we investigate a new way of generalizing this solution method. We consider different or more general classes in which we are looking for solutions.

3 A general solution procedure for first-order autonomous AODEs

In this section we introduce a procedure for finding solutions of first-order autonomous AODEs not only in the class of rational functions, but in more general classes of functions such as radical or even transcendental functions. For proofs and a detailed discussion of the procedure, we refer to Grasegger [4].

Let $F(y, y') = 0$ be an autonomous first-order AODE. We consider the corresponding algebraic curve $F(y, z) = 0$. Then obviously $\mathcal{P}_y := (y(t), y'(t))$ (for a solution y of the AODE) is a parametrization of F.

Now we take an arbitrary parametrization $\mathcal{P}(t) = (r(t), s(t))$ of F, i. e. functions r and s not both constant such that $F(r(t), s(t)) = 0$. We define $A_{\mathcal{P}}(t) := \frac{s(t)}{r'(t)}$. If it is clear which parametrization is considered, we simply write A. Assume the parametrization is of the form

$$\mathcal{P}_g = (r(t), s(t)) = (y(g(t)), y'(g(t))),$$

for unknown y and g. In this case, $A_{\mathcal{P}_g}$ turns out to be $1/g'(t)$. If we could find g (an integration problem), and its inverse g^{-1}, we also find y:

$$y(x) = r(g^{-1}(x)).$$

So we can determine a solution if we can solve the corresponding integration and function inversion problems.

Kamke [9] already mentions this procedure where he restricts to continuously differential functions r and s which satisfy $F(r(t), s(t)) = 0$. However, he gives no indication how to determine these functions.

In general g is not a bijective function. Hence, when we talk about an inverse function we actually mean one branch of a multivalued inverse. Each branch inverse will give us a solution of the differential equation.

We might add any constant c to the solution of the indefinite integral. Assume $g(t)$ is a solution of the integral and g^{-1} its inverse. Then $\bar{g}(t) = g(t) + c$ is also a solution and $\bar{g}^{-1}(t) = g^{-1}(t - c)$. We know that if $y(x)$ is a solution of the AODE, so is $y(x + c)$. Hence, we may postpone the introduction of c to the end of the procedure.

The procedure finds a solution if we can compute the integral and the inverse function. On the other hand, it does not give us any clue on the existence of a solution in case either part does not work. Neither do we know whether we have found all solutions.

3.1 Rational solutions

Feng and Gao [2] found an algorithm for computing all rational general solutions of an autonomous first order AODE. They use the fact that $(y(x), y'(x))$ is a proper rational parametrization. The main part of their algorithm says that there is a rational general solution if and only if for any proper rational parametrization $\mathcal{P}(t) = (r(t), s(t))$ we have that $A_{\mathcal{P}}(t) = q \in \mathbb{Q}$ or $A_{\mathcal{P}}(t) = a(t-b)^2$ with $a, b \in \mathbb{Q}$. The solutions therefore are $r(q(x+c))$ or $r(b - \frac{1}{a(x+c)})$ respectively.

It can be easily shown that our procedure coincides with the results from Feng and Gao. As mentioned above, our procedure does not give an answer to the question whether the AODE has a rational solution in case A is not of this special form. It might, however, find non-rational solutions for some AODEs. Nevertheless, Feng and Gao [2] already proved that there is a rational general solution if and only if A is of the special form mentioned above and all rational general solutions can be found by the algorithm.

3.2 Radical solutions

Now we extend our class of possible parametrizations and also the class in which we are looking for solutions to functions including radical expressions.

The research area of radical parametrizations is rather new. Sendra and Sevilla [17] recently published a paper on parametrizations of curves using radical expressions. In this paper Sendra and Sevilla define the notion of radical parametrization and provide algorithms to find such parametrizations in certain cases which include, but are not restricted to, curves of genus less or equal 4. Every rational parametrization will be a radical one, but obviously not the other way around. Further considerations of radical parametrizations can be found in Schicho and Sevilla [16] and Harrison [5]. There is also a paper on radical parametrization of surfaces by Sendra and Sevilla [18]. Nevertheless, for the beginning we will restrict to the case of first-order autonomous equations and hence to algebraic curves.

Definition 3.1.
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A field extension $\mathbb{K} \subseteq \mathbb{L}$ is called a radical field extension *iff $\mathbb{L}$ is the splitting field of a polynomial of the form $x^k - a \in \mathbb{K}[x]$, where k is a positive integer and $a \neq 0$. A* tower of radical field extensions *of $\mathbb{K}$ is a finite sequence of fields*

$$\mathbb{K} = \mathbb{K}_0 \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \ldots \subseteq \mathbb{K}_m$$

such that for all $i \in \{1, \ldots, m\}$, the extension $\mathbb{K}_{i-1} \subseteq \mathbb{K}_i$ is radical.

A field $\mathbb{E}$ is a radical extension field *of $\mathbb{K}$ iff there is a tower of radical field extensions of $\mathbb{K}$ with $\mathbb{E}$ as its last element.*

A polynomial $h(x) \in \mathbb{K}[x]$ is solvable by radicals *over $\mathbb{K}$ iff there is a radical extension field of $\mathbb{K}$ containing the splitting field of h.*

Let now $\mathcal{C}$ be an affine plane curve over $\mathbb{K}$ defined by an irreducible polynomial $f(x,y)$. According to [17], $\mathcal{C}$ is parametrizable by radicals *iff there is a radical extension field $\mathbb{E}$ of $K(t)$ and a pair $(r(t), s(t)) \in \mathbb{E}^2 \backslash \mathbb{K}^2$ such that $f(r(t), s(t)) = 0$. Then the pair $(r(t), s(t))$ is called a* radical parametrization *of the curve $\mathcal{C}$.*

We call a function $f(x)$ over $\mathbb{K}$ a radical function *if there is a radical extension field of $\mathbb{K}(x)$ containing $f(x)$. Hence, a* radical solution *of an AODE is a solution that is a radical function. A* radical general solution *is a general solution which is radical.*

Computing radical parametrizations as in [17] goes back to solving algebraic equations of degree less or equal 4. Depending on the degree we might therefore get more than one solution to such an equation. Each solution yields one branch of a parametrization. Therefore, we use the notation $a^{\frac{1}{n}}$ for any n-th root of a.

We will now see that the procedure mentioned above yields information about solvability in some cases.

Theorem 3.2.
Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y,z) = 0$. Assume $A_{\mathcal{P}}(t) = a(b+t)^n$ for some $n \in \mathbb{Q} \backslash \{1\}$.
Then $r(h(t))$, with $h(t) = -b + (-(n-1)a(t+c))^{\frac{1}{1-n}}$, is a radical general solution of the AODE $F(y,y') = 0$.

For the proof we refer to [4]. The algorithm of Feng and Gao is therefore a special case of this one with $n = 0$ or $n = 2$ and a rational parametrization. In exactly these two cases, g^{-1} (in the setting of Theorem 3.2) is a rational function. Furthermore, Feng and Gao [2] showed that all rational solutions can be found like this, assuming the usage of a rational parametrization. The existence of a rational parametrization is of course necessary to find a rational solution. However, in the procedure we might use a radical parametrization of the same curve which is not rational and we can still find a rational solution.

In Theorem 3.2 $n = 1$ is excluded because in this case the function g contains a logarithm and its inverse an exponential term.

Example 3.3.
The equation $y^5 - y'^2 = 0$ gives rise to the radical parametrization $\left(\frac{1}{t}, -\frac{1}{t^{5/2}}\right)$ with corresponding $A(t) = \frac{1}{\sqrt{t}}$. We can compute $g(t) = \frac{2t^{3/2}}{3}$ and $g^{-1}(t) = \left(\frac{3}{2}\right)^{2/3} t^{2/3}$. Hence, $\frac{\left(\frac{2}{3}\right)^{2/3}}{(x+c)^{2/3}}$ is a solution of the AODE.

As a corollary of Theorem 3.2 we get the following statement for AODEs where the parametrization yields another form of $A(t)$.

Corollary 3.4.
Assume we have a radical parametrization $\mathcal{P}(t) = (r(t), s(t))$ of an autonomous curve $F(y,z) = 0$ and assume $A(t) = \frac{a(b+t^k)^n}{kt^{k-1}}$ with $k \in \mathbb{Q}$. Then the AODE has a radical solution.

In contrast to the rational case there are more possible forms for A now. In the following we will see another rather simple form of A which might occur. Here we do not know immediately whether or not the procedure will lead to a solution.

Theorem 3.5.
Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$. Assume $A(t) = \frac{at^n}{b+t^m}$ for some $a, b \in \mathbb{Q}$ and $m, n \in \mathbb{Q}$ with $m \neq n-1$ and $n \neq 1$. Then the AODE $F(y, y') = 0$ has a radical solution if the function

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right) \tag{1}$$

has a radical inverse $h(x)$. A general solution of the AODE is then $r(h(x+c))$.

In case we have $n = 1$ or $m = n - 1$ the integral is a function containing a logarithm and the inverse function yields an expression containing the Lambert W-Function.

Example 3.6.
The equation $-y^5 - y' + y^8 y' = 0$ gives rise to the radical parametrization $\mathcal{P}(t) = \left(\frac{1}{t}, \frac{t^3}{1-t^8}\right)$ with corresponding $A(t) = \frac{t^5}{-1+t^8}$. Then equation (1) has a solution, e.g. $-\left(2t - \sqrt{-1+4t^2}\right)^{1/4}$. Hence, $-\left(2(x+c) - \sqrt{(-1+4(x+c)^2}\right)^{-1/4}$ is a solution of the AODE.

It remains to show when g as in (1) has an inverse expressible by radicals. Based on results of Ritt [14] on function decomposition, Grasegger shows in [4] that the following holds.

Theorem 3.7.
Assume $1 - n = \frac{z_1}{d_1}$ and $m - n + 1 = \frac{z_2}{d_2}$ with $z_1, z_2 \in \mathbb{Z}$, $d_1, d_2 \in \mathbb{N}$ such that $\gcd(z_1, d_1) = \gcd(z_2, d_2) = 1$. Let $\bar{n} = \frac{(1-n)d_1 d_2}{d}$, $\bar{m} = \frac{(m-n+1)d_1 d_2}{d}$ and $d = \gcd(z_1 d_2, z_2 d_1)$.
The function $g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n}\right)$ from Theorem 3.5 has an inverse expressible by radicals if

- *$b = 0$ or*
- *$\bar{m}, \bar{n} \in \mathbb{Z}$ and $\max(|\bar{m}|, |\bar{n}|) \leqslant 4$.*

It has no inverse expressible by radicals in the cases

- *$\bar{m}, \bar{n} \in \mathbb{N}$ and $\max(\bar{m}, \bar{n}) > 4$,*
- *$-\bar{m}, -\bar{n} \in \mathbb{N}$ and $\max(|\bar{m}|, |\bar{n}|) > 4$.*

Hence, in some cases we are able to decide the solvability of an AODE with properties as in Theorem 3.5. Nevertheless, the procedure is not complete, since even Corollary 3.7 does not cover all possible cases for m and n.

3.3 Non-Radical solutions

The procedure is not restricted to the radical case but might also solve some AODEs with non-radical solutions.

Example 3.8.
Consider the equation $y^3+y^2+y'^2=0$. *The corresponding curve has the parametrization* $\mathcal{P}(t)=(-1-t^2, t(-1-t^2))$. *We get* $A(t)=\frac{1}{2}(1+t^2)$ *and hence,* $g(t)=\int \frac{1}{A(t)}dt = 2\arctan(t)$. *The inverse function is* $g^{-1}(t)=\tan(\frac{t}{2})$ *and hence,* $y(x)=-1-\tan(\frac{x+c}{2})^2$ *is a solution.*

Beside trigonometric solutions we might also find exponential solutions.

Example 3.9.
Consider the AODE $y^2+y'^2+2yy'+y=0$. *We get the rational parametrization* $(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2})$. *With* $A(t)=-\frac{1}{2}t(1+t)$, *we compute* $g(t)=-2\log(t)+2\log(1+t)$ *and hence* $g^{-1}(t)=\frac{1}{-1+\mathrm{e}^{t/2}}$, *which leads to the solution* $-\mathrm{e}^{-(x+c)}(-1+\mathrm{e}^{(x+c)/2})^2$.

We see that it is not even necessary to use radical parametrizations in order to find non-radical solutions.

3.4 Comparison

In many books on differential equations, we can find a method for transforming an autonomous ODE of any order $F(y,y',\ldots,y^{(n)})=0$ to an equation of lower order by substituting $u(y)=y'$ and solving $\int \frac{1}{u(y)}dy = x$ for y (see for instance [22, 9]). For the case of first order ODEs, this method yields a solution of the ODE. It turns out that in this case the method is a special case of our procedure, which uses a particular kind of parametrization $\mathcal{P}(t)=(t,s(t))$.

We will now give some arguments concerning the possibilities and benefits of the general procedure. Since in the procedure any radical parametrization can be used, we might take advantage of picking a good one as we will see in the following example.

Example 3.10.
We consider the AODE $y'^6+49yy'^2-7$ *and find a parametrization of the form* $(t,s(t))$:

$$\left(t, \frac{\sqrt{\left(756+84\sqrt{28812t^3+81}\right)^{2/3}-588t}}{\sqrt{6}\left(756+84\sqrt{28812t^3+81}\right)^{1/6}}\right).$$

Neither Mathematica 8 *nor* Maple 16 *can solve the corresponding integral explicitly and hence, the procedure stops. Neither of them is capable of solving the differential equation* *in explicit form by the built in functions for solving ODEs. Nevertheless, we can try our procedure using other parametrizations. An obvious one to try next is*

$$(r(t), s(t)) = \left(-\frac{-7+t^6}{49t^2}, t\right).$$

For this parametrization both, the integral and the inverse function, are computable and the procedure yields the general solution

$$y(x) = -\frac{4\left(-7+\frac{1}{64}\left(-147(c+x)-\sqrt{7}\sqrt{32+3087(c+x)^2}\right)^2\right)}{49\left(-147(c+x)-\sqrt{7}\sqrt{32+3087(c+x)^2}\right)^{2/3}}.$$

The procedure might find a radical solution of an AODE by using a rational parametrization as we have seen in Example 3.6 and 3.10. As long as we are looking for rational solutions only, the corresponding curve has to have genus zero. Now we can also solve some examples where the genus of the corresponding curve is higher than zero and hence, there is no rational parametrization. The AODE in Example 3.11 below corresponds to a curve with genus 1.

Example 3.11.
Consider the AODE $-y^3 - 4y^5 + 4y^7 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$. *We compute a parametrization and get*

$$\left(\frac{1}{t}, \frac{-4+4t^2+t^4}{t\left(4t^2-4t^4-t^6-\sqrt{-16t^4+16t^8+8t^{10}+t^{12}}\right)}\right)$$

as one of the branches. The procedure yields the general solution

$$y(x) = -\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}.$$

Again Mathematica 8 *cannot compute a solution in reasonable time and* Maple 16 *only computes constant and implicit ones.*

4 Conclusion

We have presented a new general method for determining solutions of first-order autonomous algebraic ordinary differential equations. Our method relies crucially on the availability of a parametrization of the solution surface of the given AODE. In case the parametrization is rational, and we are considering rational solution functions, our new method simply specializes to the well known algebro-geometric method. But we may also consider non-rational parametrizations and non-rational classes of solution functions, thereby drastically enlarging the applicability of our method. Currently we are able to determine solutions for various classes of solution functions, in particular radical functions. But we are still lacking a complete decision algorithm.

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