Radical Solutions of First Order Autonomous Algebraic Ordinary Differential Equations

Georg Grasegger*
Johannes Kepler University Linz
Doctoral Program Computational Mathematics
4040 Linz, Austria
georg.grasegger@risc.jku.at

ABSTRACT

We present a procedure for solving autonomous algebraic ordinary differential equations (AODEs) of first order. This method covers the known case of rational solutions and depends crucially on the use of radical parametrizations for algebraic curves. We can prove that certain classes of AODEs permit a radical solution, which can be determined algorithmically. However, this approach is not limited to rational and radical solutions of AODEs.

Categories and Subject Descriptors

G.1.7 [**Mathematics of computing**]: Ordinary differential equations; I.1.2 [**Computing methodologies**]: Symbolic and algebraic manipulation

General Terms

Algorithms, Theory

Keywords

Algebraic ordinary differential equations, algebraic curves, radical parametrizations

1. INTRODUCTION

We consider autonomous first order algebraic ordinary differential equations $F(y, y') = 0$ and use parametrizations of curves for solving them. Hubert [8] already studied solutions of AODEs of the form $F(x, y, y') = 0$. She gives a method for finding a basis of the general solution of the equation by computing a Gröbner basis of the prime differential ideal of the general component. The solutions, however, are given implicitly. Later Feng and Gao [2, 3] started using parametrizations for solving first order autonomous AODEs, i. e. $F(y, y') = 0$. They provide an algorithm to actually find all rational solutions of such AODEs with coefficients in $\mathbb{Q}$ by using rational parametrizations of the algebraic curve $F(y, z) = 0$. The key fact they are proving is that any rational solution of the AODE gives a proper parametrization of the corresponding algebraic curve. On the other hand, if a proper parametrization of the algebraic curve fulfills some requirements, Feng and Gao can generate a rational solution of the AODE. From the rational solution it is then possible to create a rational general solution by shifting the variable by a constant. Finally Feng and Gao derive an algorithm to compute a rational solution of an AODE, if it exists.

By using this approach we can take advantage of the well known theory of algebraic curves and rational parametrizations (see for instance [21, 23]).

Recently Ngô and Winkler [12, 14, 13] worked on generalizing the algorithm of Feng and Gao. They considered non-autonomous AODEs $F(x, y, y') = 0$. Instead of algebraic curves as in the autonomous case, algebraic surfaces play a role there. For an algorithm to find rational parametrizations of surfaces see [17]. Using a rational parametrization of the surface Ngô and Winkler derive a special kind of system of differential equations. There exist solution methods for these so called associated systems. From a rational solution of such a system they find a rational general solution of the original differential equation in the generic case.

Aroca, Cano, Feng and Gao give in [1] a necessary and sufficient condition for an autonomous AODE to have an algebraic solution. They also provide a polynomial time algorithm to find such a solution if it exists. This solution, however, is implicit whereas here we try a different approach to find explicit solutions.

Furthermore Huang, Ngô and Winkler continue their work and consider higher order equations. A first result can be found in [7]. A generalization of Ngô and Winkler [14] to trivariate systems of ODEs can be found in [5, 6]. Ngô, Sendra and Winkler [11, 10] also consider classification of AODEs, i. e. they look for transformations of AODEs which keep the associated system invariant.

In this work we stick to the case of first order autonomous AODEs but try to extend the results to radical solutions using radical parametrizations (see Section 3.2). We do so by investigating a procedure which can be specified to the rational case (see Section 3.1). In contrast to the rational case we will not give a complete algorithm but rather a procedure which in the favorable case yields a result. If a result is computed then it will be a solution of the AODE. On the other hand the procedure might fail and then we cannot draw any conclusion on the solvability of the AODE.

*Supported by the Austrian Science Fund (FWF): W1214-N15, project DK11

ISSAC '14, July 23 - 25, 2014, Kobe, Japan.

http://dx.doi.org/10.1145/2608628.2608636.

Having computed a single non-trivial solution $y(x)$ we can get a general solution $y(x+c)$ for a constant c as shown in [1]. In Section 3.3 we give examples of non-radical solutions that can be found by the given procedure. Finally in Section 3.4 we look for advantages of the procedure and compare it to existing algorithms.

2. PRELIMINARIES

In the following we describe the necessary notations and definitions from differential algebra and algebraic geometry. Throughout the paper let $\mathbb{K}$ be a field of characteristic zero and $\bar{\mathbb{K}}$ its algebraic closure.

2.1 Solutions of AODEs

Consider the field of rational functions $\mathbb{K}(x)$. We will write $'$ for the usual derivative $\frac{d}{dx}$. Then $\mathbb{K}(x)$ is a differential field. We call $\mathbb{K}(x)\{y\}$ the ring of differential polynomials. The elements of this ring are polynomials in y and its derivatives, i. e. $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \ldots]$. An algebraic ordinary differential equation (AODE) is of the form $F(x, y, y', \ldots, y^{(n)}) = 0$ where $F \in \mathbb{K}(x)\{y\}$ and F is also a polynomial in x. The AODE is called autonomous if $F \in \mathbb{K}\{y\}$, i. e. if the coefficients of F do not depend on the variable of differentiation x. For a given AODE we are interested in deciding whether it has rational or radical solutions and, in the affirmative case, determining all of them.

Hence, we look for general solutions which are defined as follows. Let Σ be a prime differential ideal in $\mathbb{K}(x)\{y\}$. Then we call η a generic zero of Σ if for any differential polynomial P we have $P(\eta) = 0 \Leftrightarrow P \in \Sigma$. Such an η exists in a suitable extension field.

Let F be an irreducible differential polynomial of order n. Then $\{F\}$, the radical differential ideal generated by F, can be decomposed into two parts. There is one component where the separant $\frac{\partial F}{\partial y^{(n)}}$ also vanishes. This part represents the singular solutions. The component we are interested in is the one where the separant does not vanish. It is a prime differential ideal $\Sigma_F := \{F\} : \langle \frac{\partial F}{\partial y^{(n)}} \rangle$ and represents the general component (see for instance Ritt [16]). A generic zero of Σ_F is called a general solution of $F = 0$. We say it is a rational general solution if it is of the form $y = \frac{a_k x^k + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$, where the a_i and b_i are constants in some field extension of $\mathbb{K}$.

Here we consider only first order AODEs. For solving a differential equation $F(y, y') = 0$ we will look at the corresponding curve $F(y, z) = 0$ where we replace the derivative of y by a transcendental variable z.

2.2 Algebraic Curves

An algebraic curve $\mathcal{C}$ is a one-dimensional algebraic variety in the affine plane $\mathbb{A}^2$ over a field $\bar{\mathbb{K}}$, i. e. a zero set of a square-free bivariate non-constant polynomial $f \in \mathbb{K}[x, y]$, $\mathcal{C} = \{(a, b) \in \mathbb{A}^2 \mid f(a, b) = 0\}$. We call the polynomial f the defining polynomial. An important aspect of algebraic curves is their parametrizability. Consider an irreducible plane algebraic curve defined by an irreducible polynomial f. A tuple of rational functions $\mathcal{P}(t) = (r(t), s(t))$ is called a rational parametrization of the curve if $f(r(t), s(t)) = 0$ and not both $r(t)$ and $s(t)$ are constant. A parametrization can be considered as a map $\mathcal{P} : \mathbb{A} \to \mathcal{C}$ from the affine line $\mathbb{A}$ onto the curve $\mathcal{C}$. By abuse of notation we also call this map a parametrization. Later we will see other kinds of parametrizations. We call a parametrization $\mathcal{P}$ proper if it is a birational map or in other words if for almost every point (x, y) on the curve we find exactly one t such that $\mathcal{P}(t) = (x, y)$.

In the procedure we will assume that the input is a radical parametrization. The research area of radical parametrizations is rather new. Sendra and Sevilla [19] recently published a paper on parametrizations of curves using radical expressions. In this paper Sendra and Sevilla define the notion of radical parametrization and they provide algorithms to find such parametrizations in certain cases which include but are not restricted to curves of genus less or equal 4. Every rational parametrization will be a radical one but obviously not the other way round. Further considerations of radical parametrizations can be found in Schicho and Sevilla [18] and Harrison [4]. There is also a paper on radical parametrization of surfaces by Sendra and Sevilla [20]. Nevertheless, for the beginning we will restrict to the case of first order autonomous equations and hence to algebraic curves.

Definition 1. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A field extension $\mathbb{K} \subseteq \mathbb{L}$ is called a *radical field extension* iff $\mathbb{L}$ is the splitting field of a polynomial of the form $x^k - a \in \mathbb{K}[x]$, where k is a positive integer and $a \neq 0$. A *tower of radical field extensions* of $\mathbb{K}$ is a finite sequence of fields

$$\mathbb{K} = \mathbb{K}_0 \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \ldots \subseteq \mathbb{K}_m$$

such that for all $i \in \{1, \ldots, m\}$, the extension $\mathbb{K}_{i-1} \subseteq \mathbb{K}_i$ is radical.
A field $\mathbb{E}$ is a *radical extension field* of $\mathbb{K}$ iff there is a tower of radical field extensions of $\mathbb{K}$ with $\mathbb{E}$ as its last element.
A polynomial $h(x) \in \mathbb{K}[x]$ is *solvable by radicals* over $\mathbb{K}$ iff there is a radical extension field of $\mathbb{K}$ containing the splitting field of h.

Let now $\mathcal{C}$ be an affine plane curve over $\mathbb{K}$ defined by an irreducible polynomial $f(x, y)$. According to [19], $\mathcal{C}$ is *parametrizable by radicals* iff there is a radical extension field $\mathbb{E}$ of $\mathbb{K}(t)$ and a pair $(r(t), s(t)) \in \mathbb{E}^2 \setminus \mathbb{K}^2$ such that $f(r(t), s(t)) = 0$. Then the pair $(r(t), s(t))$ is called a *radical parametrization* of the curve $\mathcal{C}$.

We call a function $f(x)$ over $\mathbb{K}$ a *radical function* if there is a radical extension field of $\mathbb{K}(x)$ containing $f(x)$. Hence, a *radical solution* of an AODE is a solution that is a radical function. A *radical general solution* is a general solution which is radical.

Computing radical parametrizations as in [19] goes back to solving algebraic equations of degree less or equal four. Depending on the degree we might therefore get more than one solution to such an equation. Each solution yields one branch of a parametrization. Therefore, we use the notation $a^{\frac{1}{n}}$ for any n-th root of a.

In fact we do not need to restrict to rational or radical parametrizations. More generally a parametrization of f is a generic zero of the the prime ideal generated by f in the sense of van der Waerden [22].

3. A PROCEDURE FOR SOLVING FIRST ORDER AUTONOMOUS AODES

Let $F(y, y') = 0$ be an autonomous AODE. We consider the corresponding algebraic curve $F(y, z) = 0$. Then obviously $\mathcal{P}_y(t) := (y(t), y'(t))$ (for a non-trivial, i. e. non-constant solution y of the AODE) is a parametrization of F (not necessarily rational or radical).

For an arbitrary parametrization $\mathcal{P}(t) = (r(t), s(t))$ we define $A_{\mathcal{P}}(t) := \frac{s(t)}{r'(t)}$. If it is clear which parametrization is considered, we might also write A. Suppose we are given any parametrization $\mathcal{P}$ of F and we want to find a solution of the AODE. We assume the parametrization is of the form $\mathcal{P}_g(t) = (r(t), s(t)) = (y(g(t)), y'(g(t)))$ where g and y are unknown. This assumption is motivated by the fact that in the rational case each proper parametrization can be obtained from any other proper one by a linear rational transformation of the variable. If we can find g and especially its inverse function we also find y. For our given $\mathcal{P}_g$ we can compute $A_{\mathcal{P}_g}$ and furthermore we have

$$A_{\mathcal{P}_g}(t) = \frac{y'(g(t))}{\frac{d}{dt}(y(g(t)))} = \frac{y'(g(t))}{g'(t)y'(g(t))} = \frac{1}{g'(t)}.$$

Hence, by reformulation we get an expression for the unknown g':

$$g'(t) = \frac{1}{A_{\mathcal{P}_g}(t)}.$$

Using integration and inverse functions the procedure continues as follows

$$g(t) = \int g'(t)dt = \int \frac{1}{A_{\mathcal{P}_g}(t)}dt,$$
$$y(x) = r(g^{-1}(x)).$$

Kamke [9] already mentions such a procedure where he restricts to continuously differential functions r and s which satisfy $F(r(t), s(t)) = 0$. However, he does not mention where to get these functions from.

In general g is not a bijective function. Hence, when we talk about an inverse function we actually mean one branch of a multivalued inverse. Each branch inverse will give us a solution to the differential equation.

We might add any constant c to the solution of the indefinite integral. Assume $g(t)$ is a solution of the integral and g^{-1} its inverse. Then also $\bar{g}(t) = g(t) + c$ is a solution and $\bar{g}^{-1}(t) = g^{-1}(t - c)$. We know that if $y(x)$ is a solution of the AODE, so is $y(x + c)$. Hence, we may postpone the introduction of c to the end of the procedure.

We summarize the procedure for the case of radical parametrizations and solutions

Procedure 1. *Input: an autonomous AODE defined by $F(y, y') = 0$ and a radical parametrization $\mathcal{P}(t) = (r(t), s(t))$ of the corresponding curve.*
Output: a general solution y of F or "fail".

1. *Compute $A_{\mathcal{P}}(t) = \frac{s(t)}{r'(t)}$.*
2. *Compute $g(t) = \int \frac{1}{A_{\mathcal{P}}(t)}dt$.*
3. *Compute g^{-1}.*
4. *If g^{-1} is not radical return "fail", else return $r(g^{-1}(x + c))$.*

The procedure finds a solution if we can compute the integral and the inverse function. On the other hand it does not give us any clue on the existence of a solution in case either part does not work. Neither do we know whether we found all solutions.

3.1 Rational solutions

Feng and Gao [2] found an algorithm for computing all rational general solutions of an autonomous first order AODE. They use the fact that $(y(x), y'(x))$ is a proper rational parametrization. The main part of their algorithm says that there is a rational general solution if and only if for any proper rational parametrization $\mathcal{P}(t) = (r(t), s(t))$ we have that $A_{\mathcal{P}}(t) = q \in \mathbb{Q}$ or $A_{\mathcal{P}}(t) = a(t - b)^2$ with $a, b \in \mathbb{Q}$. The solutions therefore are $r(q(x + c))$ or $r(b - \frac{1}{a(x+c)})$ respectively.

We will show now that the algorithm accords with our procedure. Assume we are given an AODE with a proper parametrization $\mathcal{P}(t) = (r(t), s(t))$. Assume further that $A_{\mathcal{P}}(t) = q \in \mathbb{Q}$ or $A_{\mathcal{P}}(t) = a(t - b)^2$. Then we get from the procedure

$$\begin{aligned} A_{\mathcal{P}}(t) &= q, & A_{\mathcal{P}}(t) &= a(t - b)^2, \\ g'(t) &= \frac{1}{q}, & g'(t) &= \frac{1}{a(t - b)^2}, \\ g(t) &= \frac{t}{q} + c, & g(t) &= -\frac{1}{a(t - b)} + c, \\ g^{-1}(t) &= q(t - c), & g^{-1}(t) &= \frac{-1 + ab(t - c)}{a(t - c)}. \end{aligned}$$

We see that $r(g^{-1}(t))$ is exactly what Feng and Gao found aside from the sign of c. As mentioned above, our procedure does not give an answer to whether the AODE has a rational solution in case A is not of this special form. It might, however, find non rational solutions for some AODEs. Nevertheless, Feng and Gao [2] already proved that there is a rational general solution if and only if A is of the special form mentioned above and all rational general solutions can be found by the algorithm.

3.2 Radical solutions

Now we extend our set of possible parametrizations and also the set, in which we are looking for solutions, to functions including radical expressions. We will see that the procedure yields information about solvability in some cases.

Theorem 1. *Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$. Assume $A_{\mathcal{P}}(t) = a(b+t)^n$ for some $n \in \mathbb{Q} \setminus \{1\}$, $a \neq 0$.*
Then $r(h(x))$, with $h(x) = -b + (-(n - 1)a(x + c))^{\frac{1}{1-n}}$, is a radical general solution of the AODE $F(y, y') = 0$.

Proof. From the procedure we get

$$g'(t) = \frac{1}{A_P(t)} = \frac{1}{a(b + t)^n},$$
$$g(t) = \int g'(t)dt = \frac{(b + t)^{1-n}}{a(1 - n)},$$
$$g^{-1}(t) = -b + (-(n - 1)at)^{\frac{1}{1-n}}.$$

Then $y(x) = r(g^{-1}(x))$ is a solution of F. Let $h(x) = g^{-1}(x + c)$ for some constant c. Then $r(h(x))$ is a general solution of F. □

The algorithm of Feng and Gao is therefore a special case of this one with $n = 0$ or $n = 2$ using rational parametrizations. In exactly these two cases g^{-1} is a rational function. Furthermore, Feng and Gao [2] showed that all rational solutions can be found like this, assuming the usage of a rational parametrization. The existence of a rational parametrization is of course necessary to find a rational solution.

However, in the procedure we might use a radical parametrization of the same curve which is not rational and we can still find a rational solution.

In Theorem 1 $n = 1$ is excluded because in this case the function g contains a logarithm and its inverse an exponential term.

Example 1. The equation $y^5 - y'^2 = 0$ gives rise to the radical parametrization $\left(\frac{1}{t}, -\frac{1}{t^{5/2}}\right)$ with corresponding $A(t) = \frac{1}{\sqrt{t}}$. We can compute $g(t) = \frac{2t^{3/2}}{3}$ and $g^{-1}(t) = \left(\frac{3}{2}\right)^{2/3} t^{2/3}$. Hence, $\frac{\left(\frac{2}{3}\right)^{2/3}}{(x+c)^{2/3}}$ is a solution of the AODE.

As a corollary of Theorem 1 we get the following statement for AODEs where the parametrization yields another form of A.

COROLLARY 1. *Let $F(y, y') = 0$ be an autonomous AODE. Assume we have a radical parametrization $\mathcal{P}(t) = (r(t), s(t))$ of the corresponding curve $F(y, z) = 0$ and assume $A(t) = \frac{a(b+t^k)^n}{kt^{k-1}}$ with $k \in \mathbb{Q}$. Then F has a radical solution.*

PROOF. By transforming the parametrization by $f(t) = t^{1/k}$ to the radical parametrization $\bar{\mathcal{P}}(t) = (r(f(t)), s(f(t)))$ we compute

$$\begin{aligned} A_{\bar{\mathcal{P}}}(t) &= \frac{s(f(t))}{\frac{\partial}{\partial t}(r(f(t)))} = \frac{A(f(t))}{f'(t)} = \frac{a(b + f(t)^k)^n}{kf(t)^{k-1}f'(t)} \\ &= \frac{a(b + t^{\frac{k}{k}})^n}{kt^{\frac{k-1}{k}}\frac{1}{k}t^{\frac{1-k}{k}}} = a(b + t)^n, \end{aligned}$$

which can be solved by Theorem 1. $\square$

In the rational situation there were exactly two possible cases for A in order to guarantee a rational solution. Here, in contrast, there are more possible forms for A. In the following we will see another rather simple form of A which might occur. In this case we do not know immediately whether or not the procedure will lead to a solution.

THEOREM 2. *Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$. Assume $A(t) = \frac{at^n}{b+t^m}$ for some $a, b \in \mathbb{Q} \setminus \{0\}$ and $m, n \in \mathbb{Q}$ with $m \neq n-1$ and $n \neq 1$. Then the AODE $F(y, y') = 0$ has a radical solution if the equation*

$$\begin{aligned} &b(m - n + 1)h^{1-n} \\ &\qquad - (n - 1)h^{m-n+1} + (n - 1)(m - n + 1)at = 0 \qquad (1) \end{aligned}$$

has a non-zero radical solution for $h = h(t)$. A general solution of the AODE is then $r(h(x + c))$.

PROOF. The procedure yields

$$\begin{aligned} g'(t) &= \frac{1}{A_{\mathcal{P}}(t)} = \frac{b + t^m}{at^n}, \\ g(t) &= \int g'(t)dt = \frac{1}{a}t^{1-n}\left(\frac{b}{1-n} + \frac{t^m}{1+m-n}\right). \end{aligned}$$

The inverse of g can be found by solving the equation

$$\frac{1}{a}h(t)^{1-n}\left(\frac{b}{1-n} + \frac{h(t)^m}{1+m-n}\right) = t$$

for $h(t)$. By a reformulation and the assumptions for m and n this is equivalent to (1). $\square$

In case we have $n = 1$ or $m = n - 1$ the integral is not radical.

Example 2. For the AODE $-y^5 - y' + y^8y' = 0$ we compute the radical parametrization $\mathcal{P}(t) = \left(\frac{1}{t}, \frac{t^3}{1-t^8}\right)$ with corresponding $A(t) = \frac{t^5}{-1+t^8}$. Then equation (1) has a solution, e.g. $-\left(2t - \sqrt{-1 + 4t^2}\right)^{1/4}$. Hence, we get the solution of the AODE: $-\left(2(x + c) - \sqrt{(4(x + c)^2 - 1)}\right)^{-1/4}$.

It remains to show when (1) is solvable by radicals (i.e. when $g(t)$ in the proof of Theorem 2 has an inverse which is expressible by radicals). The following theorem due to Ritt [15] will help us to do so.

THEOREM 3. *A polynomial g has an inverse expressible by radicals if and only if it can be decomposed into*

- *linear polynomials,*
- *power polynomials x^n for $n \in \mathbb{N}$,*
- *Chebyshev polynomials and*
- *degree 4 polynomials.*

Certainly also polynomials of degree 2 and 3 are invertible by radicals but it can be shown, that Theorem 3 applies to them. We will show now that a certain polynomial is not decomposable into non-linear factors.

THEOREM 4. *Let $g(t) = C_1t^\alpha + C_2t^\beta \in \mathbb{K}[t]$ where $\mathbb{K}$ is a field of characteristic zero, $C_1, C_2 \in \mathbb{K} \setminus \{0\}$, $\alpha, \beta \in \mathbb{N}$, $\gcd(\alpha, \beta) = 1$ and $\beta > \alpha > 0$ and $\beta > 4$. Assume $g = f(h)$ for some polynomials f and h. Then $\deg f = 1$ or $\deg h = 1$.*

PROOF. Assume $g(t) = f(h(t))$ with $f = \sum_{i=0}^n a_ix^i$ and $h = \sum_{k=0}^m b_kx^k$ and $a_n \neq 0$, $b_m \neq 0$, $m, n > 1$. In case $b_0 \neq 0$ it follows that $g(t) = \bar{f}(\bar{h}(t))$ where $\bar{f}(t) = f(b_0 + t)$ and $\bar{h}(t) = h(t) - b_0$. Hence, without loss of generality we can assume that $b_0 = 0$ and therefore also $a_0 = 0$.

We denote the coefficient of a polynomial g of order k by $\mathrm{coef}_k(g)$. Let now $\tau \in \{1, \ldots, m\}$ such that $b_\tau \neq 0$ and $b_l = 0$ for all $l \in \{1, \ldots, \tau - 1\}$. Similarly let $\pi \in \{1, \ldots, n\}$ such that $a_\pi \neq 0$ and $a_l = 0$ for all $l \in \{1, \ldots, \pi - 1\}$. This implies that $\mathrm{coef}_l(g) = 0$ for all $l \in \{1, \ldots, \tau\pi - 1\}$ and $\mathrm{coef}_{\tau\pi}(g) = a_\pi b_\tau^\pi \neq 0$. Hence, $\alpha = \tau\pi$. Assume now that $\pi < n$. Then $\alpha = \tau\pi < m(n - 1) + l$ and therefore

$$\begin{aligned} 0 &= \mathrm{coef}_{m(n-1)+l}(g) \\ &= \mathrm{coef}_{m(n-1)+l}\left(a_n\left(\sum_{j=\tau}^m b_jx^j\right)^n\right) \\ &= na_nb_m^{n-1}b_l + \sum_{\bar{\varepsilon}\in E}\binom{n}{\varepsilon_{l+1}, \ldots, \varepsilon_m}a_n\prod_{j=l+1}^m b_j^{\varepsilon_j} \end{aligned}$$

for all $l \in \{\tau, \ldots, m - 1\}$ where $\bar{\varepsilon} = (\varepsilon_{l+1}, \ldots, \varepsilon_m)$ and

$$E = \left\{\bar{\varepsilon} \mid \sum_{k=l+1}^m \varepsilon_k = n, \sum_{k=l+1}^m k\varepsilon_k = m(n - 1) + l\right\}.$$

This yields, that $0 = \mathrm{coef}_{mn-1}(g) = na_nb_m^{n-1}b_{m-1}$, hence, $b_{m-1} = 0$. By induction it follows that $b_l = 0$ for all $l \in \{\tau, \ldots, m - 1\}$ which contradicts $b_\tau \neq 0$.

Therefore, $\tau = m$ or $\pi = n$. But then we have $m \mid \alpha$ and $m \mid \beta$ or $n \mid \alpha$ and $n \mid \beta$ which contradicts $\gcd(\alpha, \beta) = 1$ since $m, n \neq 1$. $\square$

Let us now consider the function

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right)$$

from the proof of Theorem 2. In general g is not a polynomial. Our aim is to find a radical function h with a radical inverse such that $\bar{g} = g(h)$ is a polynomial. Then g has a radical inverse if and only if $\bar{g}$ has a radical inverse.

Assume that $1-n = \frac{z_1}{d_1}$ and $m-n+1 = \frac{z_2}{d_2}$ with $z_1, z_2 \in \mathbb{Z}$, $d_1, d_2 \in \mathbb{N}$ such that $\gcd(z_1, d_1) = \gcd(z_2, d_2) = 1$. Then $g(t) = \bar{g}(h(t))$ where $h(t) = t^{\frac{d}{d_1 d_2}}$ and

$$\bar{g}(t) = \frac{1}{a} t^{\bar{n}} \left(\frac{b}{1-n} + \frac{t^{\bar{m}-\bar{n}}}{1+m-n} \right) \qquad (2)$$

with exponents $\bar{n} = \frac{(1-n)d_1 d_2}{d}$, $\bar{m} = \frac{(m-n+1)d_1 d_2}{d}$ and $d = \gcd(z_1 d_2, z_2 d_1)$. Hence, $\bar{m}$, $\bar{n}$ are integers with $\gcd(\bar{m}, \bar{n}) = 1$. The function h has an inverse expressible by radicals. If $m-n+1$, $1-n$ are positive integers, then also $\bar{m}, \bar{n}$ are positive. On the other hand if $n-m-1$, $n-1 \in \mathbb{N}$ we get a polynomial by a decomposition with $f(t) = t^{-1}$.

If not both $\bar{n}$ and $\bar{m}$ are positive and not both are negative but $|\bar{m}| + |\bar{n}| \leq 4$, computing the inverse function of $\bar{g}$ is the same as solving an equation of degree less or equal 4, which can be done by radicals.

We summarize this discussion using Theorem 3 and 4 as follows.

Corollary 2. *The function g from the proof of Theorem 2,*

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right),$$

has an inverse expressible by radicals in the following cases (where we use the notation from above):

- $b = 0$,
- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leq 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leq 4$,
- $-\bar{m}, \bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leq 4$,
- $\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leq 4$.

It has no inverse expressible by radicals in the cases

- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(\bar{m}, \bar{n}) > 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) > 4$.

Proof. In the cases where $\bar{m}$ and $\bar{n}$ have the same sign and $|\bar{m}|+|\bar{n}| > 4$ the function $\bar{g}$ as discussed above fulfills the requirements of Theorem 4. Hence, if $\bar{g} = f_1 \circ f_2$ either f_1 or f_2 is of degree one. It is not difficult to show, that the other one can neither be a power polynomial nor a Chebyshev polynomial. Hence, by the Theorem of Ritt $\bar{g}$ has no radical inverse.

The case where $b = 0$ is obvious. In all the other cases mentioned in the Theorem and not discussed so far we end up in solving an algebraic equation of degree less or equal four and hence, there is a radical inverse. □

Hence, in some cases we are able to decide the solvability of an AODE with properties as in Theorem 2. Nevertheless, the procedure is not complete, since even Corollary 2 does not cover all possible cases for m and n.

3.3 Non-Radical solutions

So far we were looking for rational and radical solutions of AODEs. However, the procedure is not restricted to the radical case but might also solve some AODEs with non-radical solutions as we can see in the following examples where trigonometric and exponential solutions are found.

Example 3. Consider the equation $y^3 + y^2 + y'^2 = 0$. The corresponding curve has the parametrization $P(t) = (-1 - t^2, t(-1-t^2))$. We get $A(t) = \frac{1}{2}(1+t^2)$ and hence, $g(t) = 2\arctan(t)$. The inverse function is $g^{-1}(t) = \tan(\frac{t}{2})$ and thus, $y(x) = -1 - \tan(\frac{x+c}{2})^2$ is a solution.

Example 4. Consider the AODE $y^2 + y'^2 + 2yy' + y = 0$. We get the rational parametrization $(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2})$. With $A(t) = -\frac{1}{2}t(1+t)$ we compute $g(t) = -2\log(t) + 2\log(1+t)$ and hence $g^{-1}(t) = \frac{1}{-1+e^{t/2}}$, which leads to the solution $-e^{-(x+c)}(-1+e^{(x+c)/2})^2$.

These examples show that we can find non-radical solutions even with rational parametrizations.

3.4 Comparison

In many books on differential equations we can find a method for transforming an autonomous ODE of any order $F(y, y', \ldots, y^{(n)}) = 0$ to an equation of lower order by substituting $u(y) = y'$ (see for instance [24, 9]). For the case of first order ODEs this method yields a solution. It turns out that this method is somehow related to our procedure. The method does the following:

- Substitute $u(y) = y'$,
- Solve $F(y, u(y)) = 0$ for $u(y)$,
- Solve $\int \frac{1}{u(y)} dy = x$ for y.

These computations are a special case of our general procedure where a specific form of parametrization is used, i. e. $\mathcal{P}(t) = (t, s(t))$.

We will now give some arguments concerning the possibilities and benefits of the general procedure. Since in the procedure any radical parametrization can be used we might take advantage of picking a good one as we will see in the following example.

Example 5. We consider the AODE $y'^6 + 49yy'^2 - 7$ and find a parametrization of the form $(t, s(t))$:

$$\left(t, \frac{\sqrt{\left(756 + 84\sqrt{28812t^3 + 81}\right)^{2/3} - 588t}}{\sqrt{6}\left(756 + 84\sqrt{28812t^3 + 81}\right)^{1/6}} \right).$$

Neither *Mathematica 8* nor *Maple 16* can solve the corresponding integral explicitly and hence, the procedure stops. Furthermore, neither of them is capable of solving the differential equation in explicit form by the built in functions for solving ODEs. Nevertheless, we can input another parametrization to our procedure. An obvious one to try next is

$$(r(t), s(t)) = \left(-\frac{-7 + t^6}{49t^2}, t \right).$$

It turns out that here we get $g(t) = \frac{2}{21t^3} - \frac{4t^3}{147}$. Its inverse can be computed $g^{-1}(t) = \frac{1}{2}\left(-147t - \sqrt{7}\sqrt{32 + 3087t^2}\right)^{1/3}$. Applying $g^{-1}(x+c)$ to $r(t)$ we get the solution

$$y(x) = \frac{7 - \frac{1}{64}\left(-147(x+c) - \sqrt{7}\sqrt{32 + 3087(x+c)^2}\right)^2}{\frac{49}{4}\left(-147(x+c) - \sqrt{7}\sqrt{32 + 3087(x+c)^2}\right)^{2/3}}.$$

The procedure might find a radical solution of an AODE by using a rational parametrization as we have seen in Example 2 and 5. As long as we are looking for rational solutions only, the corresponding curve has to have genus zero. Now we can also solve some examples where the genus of the corresponding curve is higher than zero and hence there is no rational parametrization. The AODE in Example 6 below corresponds to a curve with genus 1.

Example 6. Consider the AODE $-y^3 - 4y^5 + 4y^7 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$. We compute a parametrization and get

$$\left(\frac{1}{t}, \frac{-4 + 4t^2 + t^4}{t\left(4t^2 - 4t^4 - t^6 - \sqrt{-16t^4 + 16t^8 + 8t^{10} + t^{12}}\right)}\right)$$

as one of the branches. The procedure yields

$$A(t) = -\frac{t\left(-4 + 4t^2 + t^4\right)}{4t^2 - 4t^4 - t^6 - \sqrt{-16t^4 + 16t^8 + 8t^{10} + t^{12}}},$$

$$g(t) = \frac{2t^4 + t^6 + \sqrt{t^4\left(2 + t^2\right)^2\left(-4 + 4t^2 + t^4\right)}}{4t^2 + 2t^4},$$

$$g^{-1}(t) = -\frac{\sqrt{1 + t^2}}{\sqrt{1 + t}},$$

$$y(x) = -\frac{\sqrt{1 + c + x}}{\sqrt{1 + (c + x)^2}}.$$

Again *Mathematica 8* cannot compute a solution in reasonable time and *Maple 16* only computes constant and implicit ones.

4. CONCLUSION

We have introduced a procedure for solving autonomous first order ordinary differential equations. The procedure works in a given class of expressions if we can compute a certain integral and an inverse function in this class. In case of looking for rational solutions it does exactly what was known before. Furthermore, we have found some cases in which we find radical solutions. However, these cases are not yet complete and hence, this part is subject to further investigation. In case the procedure works, we have one or more solutions of the AODE. So far we do not know, whether we have all. Neither do we know anything about solvability of the AODE if the procedure does not work. We have seen that the choice of parametrization makes a difference. The influence of the choice of parametrization on the solvability of the integration and inversion problem is also a topic for further investigation.

5. ACKNOWLEDGMENTS

The author cordially thanks Franz Winkler and J. Rafael Sendra for their valuable support during this work.

6. REFERENCES

[1] J. M. Aroca, J. Cano, R. Feng, and X.-S. Gao. Algebraic General Solutions of Algebraic Ordinary Differential Equations. In M. Kauers, editor, *Proceedings of the 2005 international symposium on symbolic and algebraic computation (ISSAC), Beijing, China*, pages 29–36, New York, 2005. ACM Press.

[2] R. Feng and X.-S. Gao. Rational General Solutions of Algebraic Ordinary Differential Equations. In J. Gutierrez, editor, *Proceedings of the 2004 international symposium on symbolic and algebraic computation (ISSAC)*, pages 155–162, New York, 2004. ACM Press.

[3] R. Feng and X.-S. Gao. A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs. *Journal of Symbolic Computation*, 41(7):739–762, 2006.

[4] M. Harrison. Explicit solution by radicals, gonal maps and plane models of algebraic curves of genus 5 or 6. *Journal of Symbolic Computation*, 51:3–21, 2013.

[5] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Trivariate Rational Systems of Autonomous ODEs. In *Proceedings of the Fourth International Conference on Mathematical Aspects of Computer and Information Sciences (MACIS 2011)*, pages 93–100, 2011.

[6] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Trivariate Rational Differential Systems. *Mathematics in Computer Science*, 6(4):361–374, 2012.

[7] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Higher Order Algebraic ODEs. *Journal of Systems Science and Complexity*, 26(2):261–280, 2013.

[8] E. Hubert. The General Solution of an Ordinary Differential Equation. In Y. Lakshman, editor, *Proceedings of the 1996 international symposium on symbolic and algebraic computation (ISSAC)*, pages 189–195, New York, 1996. ACM Press.

[9] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen.* B. G. Teubner, Stuttgart, 1997.

[10] L. X. C. Ngô, J. R. Sendra, and F. Winkler. Birational Transformations on Algebraic Ordinary Differential Equations. Technical Report 12–18, RISC Report Series, Johannes Kepler University Linz, Austria, 2012.

[11] L. X. C. Ngô, J. R. Sendra, and F. Winkler. Classification of algebraic ODEs with respect to rational solvability. In *Computational Algebraic and Analytic Geometry*, volume 572 of *Contemporary Mathematics*, pages 193–210. American Mathematical Society, Providence, RI, 2012.

[12] L. X. C. Ngô and F. Winkler. Rational general solutions of first order non-autonomous parametrizable ODEs. *Journal of Symbolic Computation*, 45(12):1426–1441, 2010.

[13] L. X. C. Ngô and F. Winkler. Rational general solutions of parametrizable AODEs. *Publicationes Mathematicae Debrecen*, 79(3–4):573–587, 2011.

[14] L. X. C. Ngô and F. Winkler. Rational general solutions of planar rational systems of autonomous odes. *Journal of Symbolic Computation*,

46(10):1173–1186, 2011.

[15] J. F. Ritt. On algebraic functions, which can be expressed in terms of radicals. *Transactions of the American Mathematical Society*, 24:21–30, 1924.

[16] J. F. Ritt. *Differential algebra*, volume 33 of *Colloquium Publications*. American Mathematical Society, New York, 1950.

[17] J. Schicho. Rational Parametrization of Surfaces. *Journal of Symbolic Computation*, 26(1):1–29, 1998.

[18] J. Schicho and D. Sevilla. Effective radical parametrization of trigonal curves. In *Computational Algebraic and Analytic Geometry*, volume 572 of *Contemporary Mathematics*, pages 221–231. American Mathematical Society, Providence, RI, 2012.

[19] J. R. Sendra and D. Sevilla. Radical parametrizations of algebraic curves by adjoint curves. *Journal of Symbolic Computation*, 46(9):1030–1038, 2011.

[20] J. R. Sendra and D. Sevilla. First steps towards radical parametrization of algebraic surfaces. *Computer Aided Geometric Design*, 30(4):374–388, 2013.

[21] J. R. Sendra, F. Winkler, and S. Pérez-Díaz. *Rational Algebraic Curves, A Computer Algebra Approach*, volume 22 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin Heidelberg, 2008.

[22] B. L. van der Waerden. *Algebra*, volume II. Springer-Verlag, New York, 1991.

[23] R. J. Walker. *Algebraic curves. Reprint of the 1st ed. 1950 by Princeton University Press*. Springer-Verlag, New York - Heidelberg - Berlin, 1978.

[24] D. Zwillinger. *Handbook of Differential Equations*. Academic Press, San Diego, CA, third edition, 1998.