PARTITION ANALYSIS, MODULAR FUNCTIONS, AND COMPUTER ALGEBRA

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Abstract. This article describes recent developments connecting problems of enumerative combinatorics, constrained by linear systems of Diophantine inequalities, with number theory topics like partitions, partition congruences, and q-series identities. Special emphasis is put on the role of computer algebra algorithms. The presentation is intended for a broader audience; to this end, elementary introductions to notions like modular functions and to algorithmic aspects of algebra are given.

1. Introduction

As indicated by the title, this article has a relatively wide topical range which reaches from enumerative combinatorics and linear systems of Diophantine inequalities to number theoretic themes like partitions, partition congruences, and q-series identities. From the methods point of view, despite relying also on analytic concepts like modular functions, special emphasis is put on transforming the analytic framework into algebra, in particular, into computer algebra tools like the Ramanujan-Kolberg package to compute q-identities as witnesses for divisibility properties of partition numbers. The underlying mathematics of the Omega package is more on the algebraic side: semigroups, posets, etc. Omega is an implementation of MacMahon's method of partition analysis, having strong connections also to aspects of discrete geometry. The objective of this article is to provide an introduction to several recent developments and trends in these areas. The explanatory style of the exposition is chosen to attract also non-expert readers.

To illustrate the possible scope of applications, we quote a problem from Polya [21, Example 5]: "The three sides of a triangle are of lengths l, m, and n, respectively. The numbers l, m, and n are positive integers, $l \leq m \leq n$. Find the number of different triangles of the described kind for a given n. Find a general law governing the dependence of the number of triangles on n." The answer to this problem can be easily extracted from

$$\sum_{\substack{1\leq a\leq b\leq c\\ \text{s.t. } a+b>c}} x^a y^b z^c = \frac{xyz}{(1-yz)(1-xyz)(1-xyz^2)},$$

a relation which, as explained with other examples below, can be easily computed with the Omega package; see also [5]. Apart from elementary problems like this, partition analysis can be used in far more challenging contexts, for instance, as we shall see in Section 3 for the construction of combinatorial objects having modular forms as generating functions.

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As a second example consider $p(n)$, the number of partitions of n; for instance, $p(4) = 5$ since $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. In view of $p(9) = 30$, $p(14) = 135$, $p(19) = 490$, etc., Ramanujan conjectured that all these numbers are divisible by 5. Ramanujan also discovered a beautiful identity from which this divisibility is immediate:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}. \tag{1}$$

With regard to (1), Hardy [13, xxi–xxxvi] wrote, "It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities [...]; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting (1)." In this article we outline the structure of Radu's Ramanujan-Kolberg package which computes (1) and many other identities of this kind in automatic fashion.

The rest of this article is structured as follows. In Section 2 we give a brief account of MacMahon's partition analysis and of the "Omega project" devoted to its algorithmic revitalization. Section 3 illustrates the usage of the Omega package as a tool for mathematical discovery. We will see how Omega computations led to a new class of partitions ("broken diamond partitions") having generating functions in the form of quotients of Dedekind eta functions. One observes that arithmetic subsequences of coefficient sequences of such quotients satisfy various divisibility properties. Section 4 presents q-series identities (identities of "Ramanujan-Kolberg type") witnessing divisibility of this kind in a direct fashion. The rest of the article explains how such identities can be found in automatic fashion by the Ramanujan-Kolberg package. In Section 8 the steps of Radu's algorithm are presented. As a preparation, basic facts from modular functions are given in Section 5 and Section 6 (zero recognition of modular functions). Underlying algorithmics based on elementary facts from monoid theory is discussed in Section 7.

2. Partition Analysis

The beginning of the algorithmic revitalization of partition analysis, the "Omega project", is described by Andrews in [1, Sect. 2.10]: "The number of partitions of N of the form $b_1 + \cdots + b_n$ satisfying

$$\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0 \tag{2}$$

equals the number of partitions of N into odd parts each $\leq 2n-1$. This problem cried out for MacMahon's Partition Analysis [...] Given that Partition Analysis is an algorithm for producing partition generating functions, I was able to convince Peter Paule and Axel Riese to join an effort to automate this algorithm."

Note. A video account of this project of Andrews can be found in [27, 28].

To illustrate MacMahon's method, consider the partition problem constrained by (2) in the special case $n = 3$ but in a slightly more general setting: Find a "good" closed form of

$$L(x_1, x_2, x_3) := \sum_{b_1,b_2,b_3\in\mathbb{N} \text{ s.t. } 2b_3-3b_2\geq 0, b_2-2b_1\geq 0} x_1^{b_1}x_2^{b_2}x_3^{b_3}.$$

To remove the inequality constraints on the summation variables, MacMahon introduced the "Omega" operator. This operator acts on additionally introduced slack variables which in their exponents carry the inequality information.

$$L(x_1,x_2,x_3) = \underset{\geqq}{\Omega} \sum_{b_1,b_2,b_3\geq 0} \lambda_1^{2b_3-3b_2}\lambda_2^{b_2-2b_1} x_1^{b_1}x_2^{b_2}x_3^{b_3} = \underset{\geqq}{\Omega} \frac{1}{1-\frac{x_1}{\lambda_2^2}} \frac{1}{1-\frac{\lambda_2 x_2}{\lambda_1^3}} \frac{1}{1-\lambda_1^2 x_3}.$$

After geometric series summation, the original problem is transformed into the problem of eliminating the slack variables; here λ_1, λ_2 and λ_3. To this end, in [18] MacMahon compiled tables of elimination rules like

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda^2 A)\left(1-\frac{B}{\lambda^3}\right)} = \frac{1+A^2B}{(1-A)(1-A^3B^2)}$$

or

$$\underset{\geqq}{\Omega} \frac{\lambda^\delta}{(1-\lambda^2 A)\left(1-\frac{B}{\lambda^2}\right)} = \frac{1}{(1-A)(1-AB)}, \qquad \delta = 0,1.$$

With the Omega package (written in Mathematica) at hand, all these steps are executed automatically as follows.

In[1]:= << Omega.m

Omega Package by Axel Riese (in cooperation with George E. Andrews and Peter Paule) - ©RISC, JKU Linz - V 2.47

In[2]:= LCrude = OSum[$x1^{b1}$ $x2^{b2}$ $x3^{b3}$,

{2 b3 - 3 b2 $\geq$ 0, b2 - 2 b1 $\geq$ 0 , b1 $\geq$ 0}, λ]

Out[2]= $\underset{\lambda_1,\lambda_2}{\underset{\geq}{\Omega}} \frac{1}{\left(1-\frac{x1}{\lambda_2^2}\right)\left(1-\frac{\lambda_2\, x2}{\lambda_1^3}\right)\left(1-\lambda_1^2\, x3\right)}$

In[3]:= L=OR[LCrude]

Out[3]= $\frac{1+x2\, x3^2}{(1-x3)(1-x2^2\, x3^3)(1-x1\, x2^2\, x3^3)}$

In[4]:= L /. {x1->q, x2->q, x3->q}

Out[4]= $\frac{1+q^3}{(1-q)(1-q^5)(1-q^6)}$

Note. Out[4] presents the generating function for "Lecture Hall" partitions into maximally three parts.

It might be illuminating to put partition analysis into a somewhat more general context. To this end, consider the problem of solving linear Diophantine systems of inequalities, resp. equations, over non-negative integers. More concretely, given integers $a_{i,j}$ and c_k, find $b_1,\ldots,b_n \in \mathbb{N}$ such that

$$\begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ a_{2,1} & \ldots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}. \tag{3}$$

It is easy to see that exchanging "$\geq$" with "=" results in an equivalent problem. In both cases the algebraic structure of the set of solutions of the homogenous version of the problem (i.e., where the c_k are all 0) is that of an (additive) monoid. It is a well-known fact that this submonoid of $\mathbb{N}^n$ is finitely generated; for instance, see the classical book by Grace and Young where this is proved as a consequence of a version of the celebrated Hilbert Basis Theorem [12].

A connection to combinatorics is made by translating things to generating functions. Given an $m \times n$ integer matrix $A = (a_{ij})$, an integer vector $c = (c_1, \ldots, c_m)^t$, consider the non-negative integer solutions $b = (b_1, \ldots, b_n)^t$ to $A\, b \geq c$ presented in the form of a multivariate formal power series

$$L(x_1, \ldots, x_n) = \sum_{\substack{b=(b_1,\ldots,b_n)^t \in \mathbb{N}^n \\ Ab \geq c}} x_1^{b_1} \ldots x_n^{b_n}.$$

It turns out that such $L(x_1, \ldots, x_n)$ arising from linear Diophantine systems of inequalities (resp. equations) always are rational functions of the form

$$L(x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{(1 - x_1^{\alpha_1} \ldots x_n^{\alpha_n}) \ldots (1 - x_1^{\gamma_1} \ldots x_n^{\gamma_n})}$$

where $p(x_1, \ldots, x_n)$ is a polynomial in $x_1, \ldots, x_n$.

Note. The exponent vectors $(\alpha_1, \ldots, \alpha_n)^t, \ldots, (\gamma_1, \ldots, \gamma_n)^t$ consisting of non-negative integers are called the "fundamental" generators of the respective monoid of non-negative integer solutions to the associated homogeneous problem. Other generators come from the exponent vectors of the summand monomials in $p(x_1, \ldots, x_n)$.

Example. The example above in matrix form reads as

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

and according to `Out[3]` the solution monoid is generated as follows:

$$\alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}; \quad \alpha, \beta, \gamma \in \mathbb{N}, \delta \in \{0, 1\}.$$

In his pioneering book [18] MacMahon describes on more than a hundred pages how partition analysis is put into action to compute $L(x_1, \ldots, x_n)$ for a large variety of problems. In particular, he points to a complete algorithmic method, Elliott's algorithm, for executing this task mechanically. But without having a computer algebra system at hand, MacMahon had to use other methods - often combinations of ingenious manipulatorics and table look-up techniques.

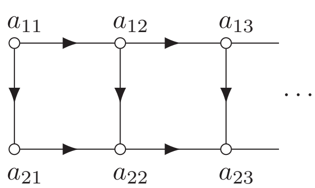
Note. MacMahon's main interest in developing partition analysis was to find a proof for his conjectured form $\prod_{n=1}^{\infty}(1 - q^n)^{-n}$ of the generating function for plane partitions. Towards the end of his exposition of partition analysis in [18] MacMahon confessed that his insight into the method was not sufficient to achieve this goal. Finally such a proof was accomplished in the course of the Omega project; see [4, 2]. Another goal of the partition analysis project was to turn algorithms into concrete software; for the resulting Omega package see [5, 6]. Also this algorithmic type of research has attracted renewed interest in the field. A very recent development is the new algorithm "Polyhedral Omega" by Breuer and Zafeirakopolous [7] which combines strategies a la MacMahon with methods from polyhedral geometry such as Brion decompositions and Barvinok's short rational function representations. In [7] one also finds careful complexity analysis and comparisons to other methods, for instance, to the work of Xin [26]. Despite the development of computer algebra packages like Omega, the primary goal of the Omega project by Andrews, Paule, and Riese was not the improvement of computational complexity but the usage of such packages in the process of mathematical discovery.

3. Omega and Mathematical Discovery

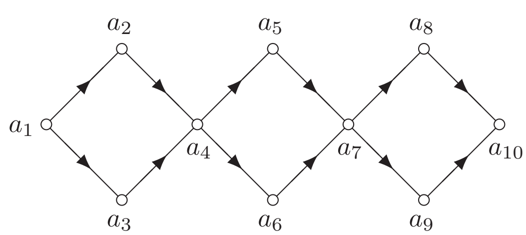
As already mentioned, MacMahon generalized partitions of numbers arranged "on a line" like $3 = 2+1 = 1+1+1$, to plane partitions arranged "in the plane", like

$$3,\ 2+1,\ \begin{array}{c}2\\+\\1\end{array},\ 1+1+1,\ \begin{array}{c}1\\+\\1\\+\\1\end{array},\ \begin{array}{l}1+1\\+\\1\end{array}.$$

Alternatively, plane partitions which, for example, are arranged in maximally two rows can be described by posets, respectively directed graphs, as follows:



Here the a_{ij} represent non-negative integers following order conditions prescribed by the arrows. For instance, the arrow from a_{11} to a_{21} means $a_{11} \geq a_{21}$. Using the Omega package it is easy to compute corresponding partition generating functions. For example, consider the following poset:

$P =$ 

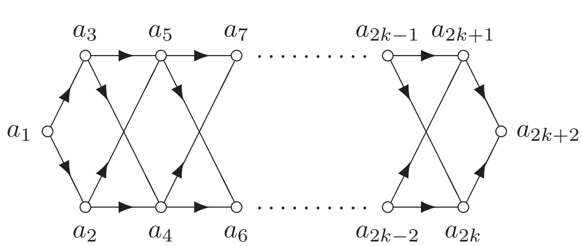
and

$$L(q) := \sum_{a_1,\ldots,a_{10}\geq 0 \text{ s.t. } P} q^{a_1+\cdots+a_{10}}.$$

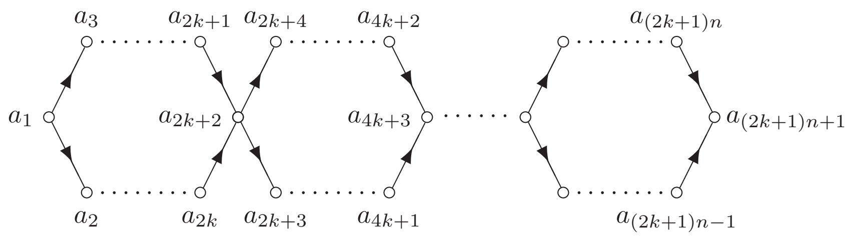
One computes the rational function presentation of $L(q)$ just as in the Omega example above, and obtains

$$L(q) = \frac{1+q^8}{(1-q)(1-q^2)^2(1-q^3)(1-q^5)^2(1-q^6)(1-q^7)(1-q^8)(1-q^9)}.$$

Computational experiments with the Omega package led to replacing the poset P by a k-elongated diamond of length 1:



More generally, one can glue n such diamonds together to obtain a k-elongated partition diamond of length n:



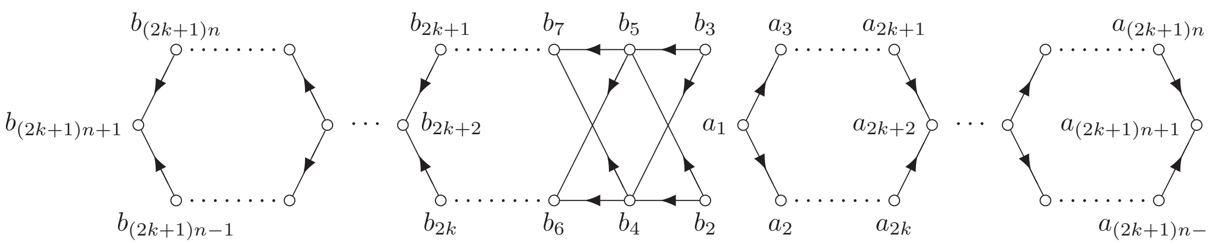
In [3] it is shown that the generating function for k-elongated diamonds of length n is

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+2})(1+q^{(2k+1)j+4})\dots(1+q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1}(1-q^j)}.$$

But one does not need to stop here. Andrews ingeniously suggested to "delete the source"; this means, to remove the a_1-vertex together with its outgoing edges. The result is a surprise; namely, for the generating function $h^*_{n,k}(q)$ over the resulting poset one obtains

$$h^*_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+1})(1+q^{(2k+1)j+3})\dots(1+q^{(2k+1)j+2k-1})}{\prod_{j=1}^{(2k+1)n}(1-q^j)}.$$

This leads us to consider the poset which results after gluing these diamonds together:



Why considering this? In the limit $n \to \infty$ the corresponding generating function becomes:

$$\begin{aligned}\sum_{m=0}^{\infty}\Delta_k(m)q^m &:= \lim_{n\to\infty} h_{n,k}(q)h^*_{n,k}(q)\\ &= \frac{\prod_{j=1}^{\infty}(1+q^j)}{\prod_{j=1}^{\infty}(1-q^j)^2\prod_{j=1}^{\infty}(1+q^{(2k+1)j})}\\ &= \frac{\prod_{j=1}^{\infty}(1+q^j)(1-q^j)}{\prod_{j=1}^{\infty}(1-q^j)^3\prod_{j=1}^{\infty}(1+q^{(2k+1)j})}\\ &= \prod_{j=1}^{\infty}\frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3(1-q^{(4k+2)j})}.\end{aligned}$$

Consequently, we have constructed combinatorial objects whose generating function is a non-trivial eta-quotient:

$$\sum_{m=0}^{\infty}\Delta_k(m)q^m = q^{\frac{k+1}{12}}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}$$

with η the Dedekind eta function defined as usual as

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{4}$$

where $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$, $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im} > 0\}$ the upper half complex plane.

Most relevant for our context, q-series defined by quotients of η-functions often posess remarkable number theoretic properties. Such properties can be studied most comfortably with the help of computer algebra systems.

For example, let us input a truncated product version of the generating function for k-elongated diamonds:

In[5]:= bd[N_, k_] :=$\prod_{j=1}^{N} \frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3(1-q^{(4k+2)j})}$

Already the case $k = 1$ will turn out to be interesting. We will inspect the coefficients of the Taylor series expansion up to that of q^{30}.

In[6]:= bd1 = Normal[Series[bd[30,1], {q,0,30}]]

Out[6]=$1 + 3q + 8q^2 + 18q^3 + 38q^4 + 75q^5 + 142q^6 + 258q^7 + 455q^8 + 780q^9 + 1308q^{10} + 2148q^{11} + 3467q^{12} + 5505q^{13} + 8168q^{14} + 13314q^{15} + 20327q^{16} + 30693q^{17} + 45882q^{18} + 67944q^{19} + 99745q^{20} + 145239q^{21} + 209882q^{22} + 301128q^{23} + 429148q^{24} + 607710q^{25} + 855414q^{26} + 1197228q^{27} + 1666585q^{28} + 2308014q^{29} + 3180668q^{30}$

First we take all the coefficients modulo 2.

In[7]:= Mod[CoefficientList[bd1, q], 2]

Out[7]= {1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0}

Next we take the coefficients modulo 3 and 4.

In[8]:= Mod[CoefficientList[bd1, q], 3]

Out[8]= {1, 0, 2, 0, 2, 0, 1, 0, 2, 0, 0, 0, 2, 0, 2, 0, 2, 0, 0, 0, 1, 0, 2, 0, 1, 0, 0, 0, 1, 0, 2}

In[9]:= Mod[CoefficientList[bd1, q], 4]

Out[9]= {1, 3, 0, 2, 2, 3, 2, 2, 3, 0, 0, 0, 3, 1, 2, 2, 3, 1, 2, 0, 1, 3, 2, 0, 0, 2, 2, 0, 1, 2, 0}

In contrast to the cases $k = 2$ and $k = 4$, a quick inspection suggests a clear pattern for $k = 3$:

(5) $$\Delta_1(2n + 1) \equiv 0 \pmod 3, \quad n \in \mathbb{N}.$$

Proof. We proceed as in [3] by recalling the freshman's dream relation

$$(1 - q^j)^3 \equiv 1 - q^{3j} \pmod 3, \quad j \in \mathbb{N}.$$

Here "$\equiv$" is considered coefficient-wise with respect to powers of q. Then

$$\sum_{m=0}^{\infty} \Delta_1(m)q^m = \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{3j})}{(1 - q^j)^3(1 - q^{6j})} \equiv \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{3j})}{(1 - q^{3j})(1 - q^{6j})} \pmod 3.$$

Hence the coefficients of odd powers of q have to be zero. □

Hirschhorn and Sellers [14] found a proof of (5) which reveals the divisibility by 3 in beautifully direct fashion. Namely, they established the identity

$$\sum_{n=0}^{\infty} \Delta_1(2n+1)q^n = 3\prod_{j=1}^{\infty} \frac{(1-q^{2j})^2(1-q^{6j})^2}{(1-q^j)^6}. \tag{6}$$

With Radu's algoithmic method [23] such identities, including (6), can be established automatically. We will discuss its underlying mathematics in the following sections.

Before doing so, we want to point out that broken diamond partition numbers satisfy a whole variety of identities similar to (5). For example, in [3] it was conjectured that for all $n \in \mathbb{N}$,

$$\Delta_2(10n+2) \equiv 0 \pmod{2} \tag{7}$$

and

$$\Delta_2(25n+14) \equiv 0 \pmod{5}. \tag{8}$$

Chan [8] proved this and also

$$\Delta_2(10n+6) \equiv 0 \pmod{2} \tag{9}$$

and

$$\Delta_2(25n+24) \equiv 0 \pmod{5}. \tag{10}$$

It should be noted that the cases $10n+2$ and $10n+6$ were first proved by Hirschhorn and Sellers [14]. Pointers to further congruences, variants and generalizations can be found in [9].

Summarizing, in this section we have seen that partition analysis can be used to construct combinatorial objects ("partition diamonds") with generating functions being eta-quotients of number theoretic interest. More precisely, subsequences of the coefficients of Taylor series expansions of such eta quotients satisfy a variety of divisibility properties. In Section 4 we shall see how such divisibilities can be explained by establishing so-called "Ramanujan-Kolberg identities" which by Radu's package can be derived automatically. In Section 5 and 6 the necessary modular function background is provided. In Section 7 we shall see that a fundamental task in Radu's algorithmic approach is to find generators of a monoid, a theme strongly related to linear Diophantine systems like (3).

4. Ramanujan-Kolberg Identities

In this section we will exemplify how proofs of congruences like (5), (7), (8), (9) and (10) can be derived automatically by using Radu's package. The main task for the package is to establish an underlying Ramanujan-Kolberg identity. Historically, this idea traces back to Ramanujan's "most beautiful" identity. More precisely, Ramanujan [24] stated without proof that for the partition numbers $p(n)$, defined by

$$\sum_{n=0}^{\infty} p(n)q^n := \prod_{j=1}^{\infty} \frac{1}{1-q^j},$$

one has (1) and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{j=1}^{\infty} \frac{(1-q^{7j})^3}{(1-q^j)^4} + 49\,q\sum_{j=1}^{\infty} \frac{(1-q^{7j})^7}{(1-q^j)^8}. \tag{11}$$

Like (6) above, these identities give a direct explanation for the first two of Ramanujan's classical congruences

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7 \tag{12}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{13}$$

For the third one, Ramanujan gave no such identity, and we will come back to this issue in Section 8.

Radu's package computes identities like (5), (1), and (11) automatically. The input-output specification of his algorithm is as follows.

INPUT: integers $\ell, m, M, N \in \mathbb{N}$ with $0 \le \ell < m$, and a sequence $(a_r(n))_{n\ge 0}$ defined with respect to a given integer tuple $r = (r_\delta)_{\delta|M}$ by

$$\sum_{n=0}^{\infty} a_r(n) q^n := \prod_{\delta|M} \prod_{n=1}^{\infty} (1-q^{\delta n})^{r_\delta}.$$

Note. In view of eta-quotients we have for $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$ that

$$\sum_{n=0}^{\infty} a_r(n) q^n = q^{-\frac{\sum_{\delta|M} \delta r_\delta}{24}} \prod_{\delta|M} \eta(\delta\tau)^{r_\delta}.$$

OUTPUT: a rational number $\sigma \in \mathbb{Q}$, an integer tuple $s = (s_\delta)_{\delta|N}$, a finite set $P_{m,r}(\ell)$ of integers such that $\ell \in P_{r,m}(\ell)$, and eta quotients $e_1, \dots, e_k$ together with $c_1, \dots, c_k \in \mathbb{Q}$ such that

$$q^\sigma \prod_{\delta|N} \prod_{n=1}^{\infty} (1-q^{\delta n})^{s_\delta} \prod_{\ell' \in P_{m,r}(\ell)} \sum_{n=0}^{\infty} a_r(mn+\ell') q^n = c_1 e_1 + \cdots + c_k e_k. \tag{14}$$

An identity of the format as in (14) is called a Ramanujan-Kolberg identity.

Note. Radu's algorithm returns such an identity only if such an identity exists.

For example, the identities (1) and (11) rewritten in this output format read as

$$q \prod_{j=1}^{\infty} (1-q^{5j}) \sum_{n=0}^{\infty} p(5n+4) q^n = 5 \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6 \tag{15}$$

and

$$q \prod_{j=1}^{\infty} (1-q^{7j}) \sum_{n=0}^{\infty} p(7n+5) q^n = 7 \left(\frac{\eta(7\tau)}{\eta(\tau)} \right)^4 + 49 \left(\frac{\eta(7\tau)}{\eta(\tau)} \right)^8, \tag{16}$$

respectively. In both cases $P_{m,r}(\ell) = \{\ell\}$ with $\ell = 4$ and $\ell = 5$, respectively.

Note. Identities of the form (14) where $P_{m,r}(\ell)$ is bigger than $\{\ell\}$ go back to Kolberg [16].

To give a concrete example in the context of broken partition diamonds, recall the congruences (8) and (10):

$$\Delta_2(25n+14) \equiv \Delta_2(25n+24) \equiv 0 \pmod 5, \quad n \in \mathbb{N}. \tag{17}$$

Using $(1-q^j)^5 \equiv 1-q^{5j} \pmod 5$ we observe that

$$\sum_{m=0}^{\infty} \Delta_2(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{5j})}{(1-q^j)^3(1-q^{10j})} \equiv \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^j)^2}{(1-q^{10j})} \pmod 5$$
$$=: \sum_{n=0}^{\infty} d(n)q^n.$$

Hence $\Delta_2(n) \equiv d(n) \pmod 5$, and we will prove (17) with $d(n)$ instead of $\Delta_2(n)$.

Radu's program "Ramanujan-Kolberg" delivers

$$q^{\frac{3}{2}} \frac{\eta(2\tau)^{12}\eta(5\tau)^{10}}{\eta(\tau)^6\eta(10\tau)^{20}} \left(\sum_{m=0}^{\infty} d(25n+14)q^n\right)\left(\sum_{m=0}^{\infty} d(25n+24)q^n\right) = 25(2t^4+28t^3+155t^2+400t+400) \tag{18}$$

where

$$t = \frac{\eta(\tau)^3\eta(5\tau)}{\eta(2\tau)\eta(10\tau)^3}. \tag{19}$$

Rewriting the η-quotients in terms of q-products of the form

$$\prod_{k=1}^{\infty}(1-q^{\delta k})^{t_k}$$

makes the divisibility by 5 for each of the two classes $d(25n+14)$ and $d(25n+24)$ explicit.

Here, in view of (14), $\ell = 14$, $m = 25$, $M = N = 10$, $r = (r_1, r_2, r_5, r_{10}) = (2, 1, 0, -1)$, $s = (s_1, s_2, s_5, s_{10}) = (-6, 12, 10, -20)$, $\sigma = -4$ and $P_{m,r}(\ell) = \{14, 24\}$.

Note. The program computes a similar identity also for $\Delta_2(n)$ instead of $d(n)$, but the output is much bigger.

For the automatic derivation of Ramanujan-Kolberg identities of the form (14) Radu has chosen a particular setting in order to deal with modular functions in an algebraic fashion. The basic ingredients to this setting are given in the Sections 5, 6, and 7. The description of the steps of Radu's Ramanujan-Kolberg Algorithm can be found in Section 8.

5. Modular Functions: Basic Notions

As defined by (4) at the end of Section 3, eta functions are holomorphic functions defined on the upper half of the complex plane. Obviously, for $f(\tau) := \eta(\tau)^{24}$ we have the periodicity $f(\tau+1) = f(\tau)$ for $\tau \in \mathbb{H}$.

Note. In view of q-series representation we recall a fundamental, but important fact. Namely, for a given holomorphic function $f(\tau)$ on $\mathbb{H}$ with period $N \in \mathbb{N} \setminus \{0\}$ (i.e., $f(\tau+N) = f(\tau)$, $\tau \in \mathbb{H}$), there exists (uniquely) a holomorphic function $h(\tau)$ on the open unit disk, punctured at 0, such that for all $\tau \in \mathbb{H}$:

$$f(\tau) = h(e^{2\pi i\tau/N}). \tag{20}$$

For example, for $f(\tau) := \eta(\tau)^{24}$ (i.e., $N = 1$) one has

$$h(q) = q\left(\sum_{n=1}^{\infty} p(n)q^n\right)^{-24}$$

for all q from the punctured open unit disc.

But more is true. Namely, $f(\tau) := \eta(\tau)^{24}$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \tau \in \mathbb{H}, \tag{21}$$

for $k = 12$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : ad - bc = 1\right\}$. Holomorphic functions on $\mathbb{H}$ with this property plus suitable asymptotic behaviour at $\tau \in \mathbb{Q} \cup \{\infty\}$ are called modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Note. A standard reference for modular forms and the related arithmetic of their q-series coefficients is [20].

Taking quotients $g(\tau) := f_1(\tau)/f_2(\tau)$ of such modular forms $f_1(\tau)$ and $f_2(\tau)$ result in stronger symmetry:

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = g(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \tau \in \mathbb{H}. \tag{22}$$

Note:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) := \frac{a\tau+b}{c\tau+d}$$

defines a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Often one writes $\gamma\tau$ instead of $\gamma(\tau)$ where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Sometimes one needs to restrict property (22) to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from subgroups of $\mathrm{SL}_2(\mathbb{Z})$, for instance, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $N \in \mathbb{N} \setminus \{0\}$, where

$$\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c\right\}.$$

Modular forms g of weight $k = 0$ with symmetry (22) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ (i.e., (23) below) and being holomorphic on $\mathbb{H}$ are called modular functions for $\Gamma_0(N)$. It is obvious that such functions for fixed N form $\mathbb{C}$-algebras; i.e., they are commutative rings with 1 and vector spaces over $\mathbb{C}$.

Notation. The $\mathbb{C}$-algebra of modular functions for $\Gamma_0(N)$ will be denoted by $M(N)$.

For example, one can show that the functions $(\eta(5\tau)/\eta(\tau))^6$ from (15), and $f(z)$ and $f(z)^2$ with $f(z) = (\eta(7\tau)/\eta(\tau))^4$ from (16) are elements from $M(5)$ and $M(7)$, respectively. To this end, one needs to verify for $N = 5$, resp $N = 7$ the following variant of (22) :

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = g(\tau), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ \tau \in \mathbb{H}, \tag{23}$$

and the suitable asymptotic behaviour at all $\tau \in \mathbb{Q}\cup\{\infty\}$. This "suitable asymptotic behaviour" is described by using the Laurent expansion as in (20) of h around 0. Consider $\tau \in \mathbb{H}$ close to ∞ or to a point $\frac{a}{c} \in \mathbb{Q}$. Allowing $c = 0$ we include the case $\infty = \frac{a}{0}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have $\gamma(\infty) = \frac{a}{c}$. Because of periodicity the following theorem holds [15, Thm 4].

Theorem 5.1. *Let* $\gamma(\infty) = \frac{a}{c}$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *and* g *be a holomorphic function on* $\mathbb{H}$ *satisfying* (23). *Then for all* $\tau \in \mathbb{H}$ *sufficiently close to* $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ *there exists a Laurent series expansion such that*

$$g(\tau) = \sum_{n=-\infty}^{\infty} c_n(\gamma) e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma} \tag{24}$$

where

$$w_\gamma := \min\Big\{ h \in \mathbb{N}^* : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_0(N)\gamma \Big\}.$$

Now we can give a precise definition of modularity. Namely, g as in the theorem is called a modular function, if (23) holds, and if

$$\text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \ c_n(\gamma) = 0 \text{ for almost all negative } n. \tag{25}$$

(Instead of (25) one also says that g is meromorphic at $\tau = \frac{a}{c} = \gamma\infty$.) If this holds and if m is the smallest index such that $c_m(\gamma) \neq 0$, then we call m the γ-order of g at $\tau = \frac{a}{c}$; notation: $m = \mathrm{ord}^{\gamma}_{a/c}(g)$.

Note. If $c_n(\gamma) = 0$ for all $n \in \mathbb{Z}$ we set $\mathrm{ord}^{\gamma}_{a/c}(g) = \infty$.

Using the fact that $\gamma_2^{-1}\gamma_1\infty = \infty$ iff $\gamma_2^{-1}\gamma_1 = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for some $k \in \mathbb{Z}$, it is not too difficult to check that if $\frac{a}{c} = \gamma_1\infty = \gamma_2\infty$ for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, then $w_{\gamma_1} = w_{\gamma_2}$ and

$$\mathrm{ord}^{\gamma_1}_{a/c}(g) = \mathrm{ord}^{\gamma_2}_{a/c}(g). \tag{26}$$

Thus we can define the order of a modular function g at $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ by

$$\mathrm{ord}_{a/c}(g) := \mathrm{ord}^{\gamma}_{a/c}(g)$$

for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = \frac{a}{c}$.

On the same line, let g be a modular function, and let $\gamma\infty = \frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with expansion at $\frac{a}{c}$ being

$$g(\tau) = \sum_{n \geq \mathrm{ord}_{a/c}} c_n(\gamma) e^{2\pi i(\gamma^{-1}\tau)/w_\gamma}.$$

Furthermore, let $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma' = \gamma_0\gamma$ for some $\gamma_0 \in \Gamma_0(N)$. Then

$$w_{\gamma'} = w_\gamma$$

owing to $(\gamma')^{-1}\Gamma_0(N)\gamma' = \gamma^{-1}\Gamma_0(N)\gamma$. In addition, g has an expansion at $\frac{a'}{c'} = \gamma'\infty$ as follows

$$\begin{aligned} g(\tau) &= \sum_{n \geq \mathrm{ord}_{a'/c'}(g)} c_n(\gamma') e^{2\pi i((\gamma')^{-1}\tau)/w_{\gamma'}} \\ &= \sum_{n \geq \mathrm{ord}_{a'/c'}(g)} c_n(\gamma') e^{2\pi i(\gamma^{-1}\tau)/w_\gamma} \end{aligned}$$

where the second equality is by $g(\tau) = g(\gamma_0\tau)$. Hence, by uniqueness of Laurent expansion, for all $n \in \mathbb{Z}$:

$$c_n(\gamma') = c_n(\gamma); \text{ in particular , } \mathrm{ord}^{\gamma'}_{a'/c'}(g) = \mathrm{ord}^{\gamma}_{a/c}(g). \tag{27}$$

In other words, the (24) expansions at points $\frac{a'}{c'} = \gamma_0 \frac{a}{c}$ with $\gamma_0 \in \Gamma_0(N)$ are all the same.

Example. For $g(\tau) := (\eta(5\tau)/\eta(\tau))^6$ property (23) for $N = 5$ follows from a refined version of the symmetry (21) for the η-function. To show property (25) we need to show that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the expansion (24) has only a finite sum as its principal part. In view of property (27), this task can be reduced to a finite number of inspections. To this end, we make use of the coset decomposition [15].

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(5) \cup \Gamma_0(5)T \cup \Gamma_0(5)TS \cup \cdots \cup \Gamma_0(5)TS^4 \tag{28}$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

this means, we just need to check for $\gamma = \{\mathrm{id}, T, TS, \ldots, TS^4\}$.

A further reduction of the number of inspections comes from the fact that $S(\infty) = \infty$ and thus

$$\mathrm{id}(\infty) = \infty \text{ and } TS^j(\infty) = 0 \ (j = 0, \ldots, 4).$$

This means, we need to inspect the Laurent expansion (24) for all τ close to ∞ (i.e., choosing $\gamma = \mathrm{id}$) and close to 0 (i.e., for $\gamma = T$). For $\gamma = \mathrm{id}$ we can invoke (4) with $q = e^{2\pi i \tau}$

$$g(\tau) = q \prod_{j=1}^{\infty} \left(\frac{1 - q^{5j}}{1 - q^j} \right)^6;$$

i.e., $\mathrm{ord}_\infty(g) = 1$. For $\gamma = T$ we have to use [15, Thm. 9]

$$\eta\Big(-\frac{1}{\tau}\Big) = (-i)^{1/2} \tau^{1/2} \eta(\tau), \ \tau \in \mathbb{H}, \tag{29}$$

(taking that branch of the square root function $\tau^{1/2}$ which is positive for real $\tau > 0$) and obtain for τ close to 0:

$$g(\tau) = \frac{1}{5^3} \frac{1}{Q} \prod_{j=1}^{\infty} \Big(\frac{1 - Q^j}{1 - Q^{5j}} \Big)^6 \tag{30}$$

where $Q = e^{2\pi i (T^{-1}\tau)/5}$. This means, $\mathrm{ord}_0(g) = -1$.

Note 1. Relation (30) corresponds to the following relation:

$$g(\tau) g\Big(-\frac{1}{5\tau}\Big) = \frac{1}{5^3}, \quad \tau \in \mathbb{H}.$$

Note 2. Matching the right hand side of (30) to the series expansion as in (24) one sees that $w_T = 5$. This is in accordance with the fact [17, Lemma 3.2.4]

$$w_\gamma = \frac{N}{\gcd(c^2, N)} \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{31}$$

Summarizing, with these considerations we have sharpened our understanding of modular functions for a given group $\Gamma_0(N)$. Recall that these objects form a $\mathbb{C}$-algebra which we denoted by $M(N)$. We also note that if $f(\tau) \in M(N)$ without having roots in the upper half complex plane, then also

$$\frac{1}{f(\tau)} \in M(N).$$

6. Modular Functions: Zero Recognition

Our task is to establish and to prove algebraic relations between functions from $M(N)$. To illustrate various points, consider the following functions on $\mathbb{H}$ using $q = e^{2\pi i \tau}$:

$$f(\tau) := q \prod_{j=1}^{\infty} (1 - q^{5j}) \sum_{n=0}^{\infty} p(5n+4) q^n \tag{32}$$

and

$$g(\tau) := q \prod_{j=1}^{\infty} \left(\frac{1-q^{5j}}{1-q^j} \right)^6 = \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6 . \tag{33}$$

Following the example above we know that $g \in M(5)$. Suppose we also know that $f \in M(5)$. Equipped with this knowledge: how does one prove (15); i.e.,

$$f(\tau) - 5g(\tau) = 0, \ \tau \in \mathbb{H}? \tag{34}$$

Since both functions can be expressed as power series in q, one way to proceed would be comparing the (infinitely) many coefficients in these expansions. However, in the given context, it is non-trivial to turn this strategy into a feasible (finitary) argument. Rather than that, one looks at expansions at points $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$, where poles arise. In view of (24), the representations in (32) and (33) correspond to expansions at $\gamma\infty = \frac{1}{0} = \infty$ with $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and where $w_\infty = 1$ by (31). Next consider the expansion at $\gamma\infty = \frac{0}{1} = 0$ with $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T$ and $w_0 = 5$ by (31). If we expand (30) in powers of $Q = e^{2\pi i (T^{-1}\tau)/5}$ we obtain

$$g(\tau) = \frac{1}{5^3} Q^{-1} - \frac{6}{5^3} + \frac{9}{5^3} Q^1 + \dots .$$

As we shall see below, to prove (15) it is sufficient to check for the Q-expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} c_n(T) Q^n,$$

whether $\mathrm{ord}_0(f) = -1$ and whether

$$c_{-1}(T) = \frac{1}{5^2} \text{ and } c_0 = -\frac{6}{5^2}. \tag{35}$$

This is done as follows. First, rewrite $f(\tau)$, as defined in (32) as

$$f(\tau) = q^{\frac{19}{24}} \eta(5\tau) \sum_{n=0}^{\infty} p(5n+4) q^n$$

and define

$$H(\tau) := \frac{1}{5} \eta(5\tau) \text{ and } F(\tau) := 5 q^{\frac{19}{24}} \sum_{n=0}^{\infty} p(5n+4) q^n.$$

By (29) we have

$$H(T\tau) = \frac{1}{5} (-i)^{1/2} \left(\frac{\tau}{5} \right)^{1/2} (q^{\frac{1}{5}})^{24} \prod_{j=1}^{\infty} (1 - (q^{\frac{1}{5}})^j).$$

To express $F(T\tau)$ in terms of powers of $q^{\frac{1}{5}}$ is more involved. Using properties of the eta function, Rademacher [22] derived that

$$\begin{aligned} F(T\tau) =& \sum_{\lambda=0}^{4} \eta\Big(\frac{\tau+24\lambda}{5}\Big)^{-1} \\ =&(-i)^{-1/2}(5\tau)^{-1/2}(q^{\frac{1}{5}})^{-\frac{25}{24}} \\ &\times \left(\prod_{n=1}^{\infty} \Big(1-(q^{\frac{1}{5}})^{25n}\Big)^{-1} - 5\sum_{n=1}^{\infty} \Big(\frac{n}{5}\Big) p(n-1)(q^{\frac{1}{5}})^{n} \right) \end{aligned} \tag{36}$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol. Consequently,

$$\begin{aligned} f(\tau) =& f(T(T^{-1}\tau)) = H(T(T^{-1}\tau))F(T(T^{-1}\tau)) \\ =& \frac{1}{5^2} Q^{-1} \prod_{j=1}^{\infty}(1-Q^j) \\ =& \times \left(\sum_{n=0}^{\infty} p(n)Q^{25n} - 5\sum_{n=1}^{\infty} \Big(\frac{n}{5}\Big) p(n-1)Q^n \right) \\ =& \frac{1}{5^2}Q^{-1} - \frac{6}{5^2} + \frac{9}{5^2}Q + \dots, \end{aligned}$$

which confirms (35). Why is this sufficient to prove (34)?

Recall the following classical fact from complex analysis:

Theorem 6.1 (MMT)**.** *Let f be a holomorphic function on a connected open subset $U \subseteq \mathbb{C}$. Suppose there is a point $p \in U$ such that $|f(z)| \leq |f(p)|$ for all $z \in U$. Then f is constant on U.*

This theorem ("Maximum Modulus Theorem") generalizes word by word replacing $\mathbb{C}$ by a connected Riemann surface X; see [19, Thm. 1.36]. As a corollary one obtains a fundamental tool for zero testing of modular functions:

Theorem 6.2 (ZT)**.** *Let f be a holomorphic function on a compact Riemann surface X. Then f is a constant function.*

Proof. The function $|f|$ is continuous on the compact space X, hence taking on a maximum at some point in X. Consequently, the Riemann surface version of the MMT implies that f is a constant function. □

Note. Taking $U = \mathbb{C}$ in MMT gives Liouville's theorem. In view of ZT this would correspond to choosing the Riemann sphere for X.

Where in our context is the compact Riemann surface X to apply theorem ZT for zero-testing of modular functions?

To answer this question we extend our group action from $\mathbb{H}$ to $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$:

$$\begin{array}{rcl} \Gamma_0(N) \times \overline{\mathbb{H}} & \to & \overline{\mathbb{H}}, \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) & \mapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \dfrac{a\tau+b}{c\tau+d}. \end{array}$$

Notation. We write $[\tau] := \{\gamma\tau : \gamma \in \Gamma_0(N)\}$ for τ-orbits and $X_0(N)$ for the set of all orbits.

We remark that orbits of $\tau \in \mathbb{Q} \cup \{\infty\}$ contain only elements from $\mathbb{Q} \cup \{\infty\}$. Thus

$$X_0(N) = \{[\tau] : \tau \in \mathbb{H}\} \cup \{[\tau] : \tau \in \mathbb{Q} \cup \{\infty\}\}$$

as a disjoint union of orbit sets.

Note. There are only finitely many orbits in $\{[\tau] : \tau \in \mathbb{Q} \cup \{\infty\}\}$. These orbits are called cusps of $X_0(N)$; the underlying geometrical motivation and the connection to Riemann surfaces can be found in books like [10]. Sometimes, by abuse of language, also a representative τ of a cusp $[\tau]$ is called cusp.

Example. For $N = 5$ we determine $\{[\tau] : \tau \in \mathbb{Q} \cup \{\infty\}\}$: Either $\tau = \frac{a}{c} = \frac{1}{0} = \infty$ or $\tau = \frac{a}{c}$ for relatively prime integers a and c. In any case there exists $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = \frac{a}{c}$. According to (28), $\gamma \in \Gamma_0(5)$ or $\gamma = \gamma_0 T S^j$ with $\gamma_0 \in \Gamma_0(5)$ and $j \in \{0, \dots, 4\}$. In the first case $\left[\frac{a}{c}\right] = [\infty]$, in the second case $\left[\frac{a}{c}\right] = [0]$ because of $TS^j\infty = 0$. Hence

$$X_0(5) = \{[\tau] : \tau \in \mathbb{H}\} \cup \{[0], [\infty]\}.$$

Modular functions on $\mathbb{H}$ can be turned into meromorphic functions on $X_0(N)$ as follows: Suppose $f \in M(N)$. For $\tau \in \mathbb{N}$ define

$$\tilde{f}([\tau]) := f(\tau).$$

Because of (23) this is well defined. For $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ let $\gamma\infty = \frac{a}{c}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Consider the Laurent expansion as in (24) for $\tau \in \mathbb{H}$ close to $\frac{a}{c}$:

$$f(\tau) = \sum_{n=\mathrm{ord}_{a/c}(f)}^{\infty} c_n(\gamma) e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma}$$

with $w_\gamma = \frac{N}{\gcd(c^2, N)}$ as in (31). If now $\mathrm{ord}_{a/c}(f) < 0$, define

$$\tilde{f}\left(\left[\frac{a}{c}\right]\right) := \infty.$$

If $\mathrm{ord}_{a/c} = 0$, define

$$\tilde{f}\left(\left[\frac{a}{c}\right]\right) := c_0(\gamma);$$

if $\mathrm{ord}_{a/c}(f) > 0$, define

$$\tilde{f}\left(\left[\frac{a}{c}\right]\right) := 0.$$

Because of (27) all these function values at $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ are well-defined. Consequently, with the definitions above any $f \in M(N)$ gives rise to a function $\tilde{f}$ on $X_0(N)$.

Without going into detail, the set of orbits $X_0(N)$ can be equipped with a (natural) topology to make it a compact Hausdorff space. In addition, by introducing suitable charts (local homeomorphisms from $X_0(N)$ to $\mathbb{C}$) $X_0(N)$ can be turned into a Riemann surface.

Therefore, modular functions $f \in M(N)$ can be viewed as functions $\tilde{f}$ on the compact Riemann surface $X_0(N)$. One can check in a straight-forward fashion that the functions $\tilde{f}$ are meromorphic functions on $X_0(N)$ which, owing to f being holomorphic on $\mathbb{H}$, have possible poles only at the cusps; i.e., at the points $\left[\frac{a}{c}\right]$ with $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$.

As concrete representations at these cusps, one has that for all $\tau \in \mathbb{H}$ from a suitably chosen open neighbourhood of $\frac{a}{c} = \gamma\infty \in \mathbb{Q} \cup \{\infty\}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

$$\tilde{f}([\tau]) = f(\tau) = \sum_{n=\mathrm{ord}_{a/c}}^{\infty} c_n(\gamma) e^{2\pi i(\gamma^{-1}\tau)/w_\gamma}.$$

Note. As "suitable neighbourhoods" of $\frac{a}{c}$ for $\frac{a}{c} \in \mathbb{Q}$ one can take open discs in $\mathbb{H}$ which are tangent to the real axis and to which $\frac{a}{c}$ is adjoined; for $\frac{a}{c} = \infty$ one can choose the "degenerated discs" $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > c\}$, $c \geq 0$.

If $\mathrm{ord}_{a/c} \geq 0$ for all cusps $\left[\frac{a}{c}\right]$ then $\tilde{f}$ is holomorphic and, by Theorem ZT, $\tilde{f}$ is even a constant function on $X_0(N)$.

As an obvious consequence, f is constant function on $\mathbb{H}$. And as another obvious consequence we obtain the following zero test for a modular function $f \in M(N)$:

MF-ZeroTest.

(T1) Determine all different cusps $\{\left[\frac{a}{c}\right] : \frac{a}{c} \in \mathbb{Q} \cup \{\infty\}\}$.

(T2) For each cusp representative $\frac{a}{c} = \gamma\infty$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ determine whether all the coefficients of the principal part

$$\sum_{n=\mathrm{ord}_{a/c}(f)}^{-1} c_n(\gamma) e^{2\pi i(\gamma^{-1}\tau)/w_\gamma}$$

are zero.

(T3) If the answer to (T2) in each instance is yes: choose a suitable cusp $\left[\frac{a}{c}\right]$ to test whether $c_0(\gamma) = 0$.

7. Interlude: Monoids and Modular Functions

Before describing the Ramanujan-Kolberg Algorithm in Section 8, we have to introduce a special $\mathbb{C}$-algebra of modular functions. This algebra and a related (commutative) monoid structure allow to simplify the MF-ZeroTest and to carry out also other algebraic/algorithmic tasks.

To this end, we consider those functions $f \in M(N)$ for which $\tilde{f}$ as a meromorphic function on $X_0(N)$ has a pole only at the cusp $\left[\frac{a}{c}\right] = [\infty]$; i.e.,

$$M^\infty(N) := \{f \in M(N) : \mathrm{ord}_{a/c}(f) \geq 0 \text{ for all } \frac{a}{c} \in \mathbb{Q}\}.$$

As its ambient space $M(N)$, also $M^\infty(N)$ forms a $\mathbb{C}$-algebra which, in addition, gives rise to a naturally defined additive monoid induced by the pole orders.

The natural numbers $\mathbb{N} = \{0, 1, \dots\}$ form a commutative monoid with respect to both multiplication (with identity element 1) and addition (with identity element 0). Let $M \subseteq \mathbb{N}$ be an additive submonoid. It is easily seen that M is finitely generated. Namely, fix some $m \in M \setminus \{0\}$ and consider in M the equivalence classes modulo m. If $a, a' \in M$ such that $a' = a + \ell m$ for some $\ell \in \mathbb{N}$, then a' can be discarded as a generator. Hence, choosing the minimal element $u_i \in M$ from each equivalence class gives a set of generators $\{u_0 = 0, u_1, \dots, u_{k-1}, m\}$, $k \leq m$, that generate all of M.

Note. It may happen that $k < m$ because some of the equivalence classes may have empty intersection with M. However, this cannot happen in the case of numerical semigroups M which have the additional defining property that $1 \in G_M$ (i.e., $G_M = \mathbb{Z}$) where

$$G_M := \{\alpha_1 m_1 + \cdots + \alpha_\ell m_\ell : \ell \in \mathbb{N}, \alpha_i \in \mathbb{Z}, m_j \in M\}$$

is the (cyclic) subgroup of $\mathbb{Z}$ generated by M. Because then one can find a representation of 1 as

$$1 + \alpha_1 m_1 + \cdots + \alpha_i m_i = \alpha_{i+1} m_{i+1} + \cdots + \alpha_j m_j \in M$$

with $\alpha_k \in \mathbb{N}$. For $m' := \alpha_1 m_1 + \cdots + \alpha_i m_i \in M$ we have $1 + m' \in M$; this implies $u_i := i(1 + m') \in M$ are solutions of the congruences $u_i \equiv i \pmod{m'}$ for each $i \in \{0, \ldots, m'-1\}$. Hence $\mathbb{N} \setminus M$ is a finite set, and also modulo any other non-zero element $m \in M$ no congruence class in M can remain empty. Vice versa, if $\mathbb{N} \setminus M$ is a finite set for a submonoid $M \subseteq \mathbb{N}$ then there exists an $m' \in M$ such that $1 + m' \in M$; hence $(1 + m') - m' = 1 \in G_M$ and M is a numerical semigroup. Finally we note that considering numerical semigroups is no serious restriction of generality because for each submonoid M of $\mathbb{N}$ there is a monoid homomorphism $\phi : M \to M_d, m \mapsto m/d$, where $M_d := \{m/d : m \in M\}$ is a numerical semigroup with d being the generator of G_M. Further information on numerical semigroups with connections to Diophantine systems of inequalities like (3) can be found in [25].

Given non-constant modular functions $f, f_1, \ldots, f_n \in M^\infty(N)$, our goal is to develop an algorithm for checking membership in the $\mathbb{C}$-algebra R generated by $\{f_1, \ldots, f_n\}$; i.e., to decide whether

$$f \in R := \mathbb{C}[f_1, \ldots, f_n]?$$

For simplicity we assume $\gcd(\operatorname{ord}_\infty(f_1), \ldots, \operatorname{ord}_\infty(f_n)) = 1$. In this case

$$M := \{-\operatorname{ord}_\infty(g) : g \in R\}$$

is a numerical semigroup. Define $m := -\operatorname{ord}_\infty(f_1)$ and $t := f_1$, and let $u_0 = 0$, $u_1, \ldots, u_{m-1}$, and m be the generators of M as explained above. Let $g_0 = 1$, $g_1, \ldots, g_{m-1} \in R$ be such that $-\operatorname{ord}_\infty(g_i) = u_i$ for $i \in \{0, \ldots, m-1\}$, then we have

Theorem 7.1. *For every $f \in R$ there exist polynomials $p_0(x), \ldots, p_{m-1}(x) \in \mathbb{C}[x]$ such that*

$$f = p_0(t) + p_1(t)\, g_1 + \cdots + p_{m-1}(t)\, g_{m-1}.$$

Proof. If $d = \operatorname{ord}_\infty(f)$, then $d = u_i + \ell m$ for some index $i \in \{0, \ldots, m-1\}$ and some $\ell \in \mathbb{N}$. In particular, $\operatorname{ord}_\infty(g_i t^\ell) = \operatorname{ord}_\infty(f)$; therefore there exists $c \in \mathbb{C}$ such that $\operatorname{ord}_\infty(f - c g_i t^\ell) > \operatorname{ord}_\infty(f)$. Continuing this process with $h_1 := f - c g_i t^\ell \in R$ instead of f, one obtains an $h_2 \in R$, and so on. Finally, after k steps one obtains an $h_k \in R$ with $\operatorname{ord}_\infty(h_k) > 0$, and thus $h_k = 0$ by the MF-ZeroTest. By back substitution we obtain the desired form of f. □

Theorem 7.2. *With the assumptions as in the previous theorem we have that*

$$\mathbb{C}[f_1, \ldots, f_n] = \langle g_0 = 1, g_1, \ldots, g_{m-1} \rangle_{\mathbb{C}[t]}, \tag{37}$$

the right side denoting the $\mathbb{C}[t]$-module generated by $g_0 = 1, g_1, \ldots, g_{m-1} \in M^\infty(N)$ where the g_i are computed according to the Algorithm MODULE GENERATORS.

Proof. The following algorithm description shows how to obtain $g_1, \ldots, g_{m-1}$ from $f_1, \ldots, f_n$ in an algorithmic fashion. The rest of the statement is only a reformulation of Theorem 7.1. □

Algorithm MODULE GENERATORS.

INPUT: non-constant $t := f_1, f_2, \dots, f_n \in M^\infty(N)$ with $m := -\operatorname{ord}_\infty(t)$ and

$$\gcd(\operatorname{ord}_\infty(t), \operatorname{ord}_\infty(f_2), \dots, \operatorname{ord}_\infty(f_n)) = 1;$$

OUTPUT: $g_0 = 1, g_1, \dots, g_{m-1} \in M^\infty(N)$ such that (37).

STEP 0: Set $E := \{\}$. Set $m_k := -\operatorname{ord}_\infty(f_k)$, $k = 2, \dots, n$. For each $1 \le i \le m-1$: if there exists $f_j \in \{f_2, \dots, f_n\}$ such that $-\operatorname{ord}_\infty(f_j) \equiv i \pmod m$ then do nothing. Suppose such f_j for i does not exist. Then there exist $\alpha_{i,k} \in \mathbb{N}$ such that

$$\alpha_{i,2} m_2 + \alpha_{i,3} m_3 + \dots \alpha_{i,n} m_n \equiv i \pmod m$$

owing to $\gcd(m, m_2, m_3, \dots, m_n) = 1$; in this case, add to the set E the element $f_2^{\alpha_{i,2}} f_3^{\alpha_{i,3}} \cdots f_n^{\alpha_{i,n}}$. — Remark. After STEP 0 the set $\{h_1, \dots, h_\ell\} := \{f_2, \dots, f_n\} \cup E$ has the property

$$\{-\operatorname{ord}_\infty(h_1), \dots, -\operatorname{ord}_\infty(h_\ell)\} \equiv \{1, \dots, m-1\} \pmod m.$$

STEP 1: Define as input list $L := \{h_1, \dots, h_\ell\} := \{f_2, \dots, f_n\} \cup E$. To do an "elementary step" consider a pair $\{h_i, h_j\} \subseteq L$ with $-\operatorname{ord}_\infty(h_j) \ge -\operatorname{ord}_\infty(h_i)$. If

$$-\operatorname{ord}_\infty(h_j) \not\equiv -\operatorname{ord}_\infty(h_i) \pmod m,$$

do nothing; otherwise, set

$$F := h_j - c\, t^{\frac{-\operatorname{ord}_\infty(h_j) + \operatorname{ord}_\infty(h_i)}{m}} h_i$$

where $c \in \mathbb{C}$ is chosen such that $-\operatorname{ord}_\infty(F) < -\operatorname{ord}_\infty(h_j)$. If $F = 0$, delete h_j from L; otherwise, replace h_j by F.

Repeat this "elementary step" until the list of functions remains unchanged.

Note 1. STEP 1 turns L into $\{F_1, \dots, F_{m-1}\} \subseteq M^\infty(N)$ such that

$$\mathbb{C}[f_1, f_2, \dots, f_n] = \mathbb{C}[t, f_2, \dots, f_n] = \mathbb{C}[h_1, h_2, \dots, h_\ell] = \mathbb{C}[t, F_1, \dots, F_{m-1}]$$

and

$$\{-\operatorname{ord}_\infty(F_1), \dots, -\operatorname{ord}_\infty(F_{m-1})\} \equiv \{1, \dots, m-1\} \pmod m.$$

Note 2. In view of the finiteness of $\mathbb{N} \setminus M'$ the monoid M' generated by the orders $-\operatorname{ord}_\infty(F_1), \dots, -\operatorname{ord}_\infty(F_{m-1})$ is a numerical semigroup. But, in general, M' is only a submonoid of $M = \{-\operatorname{ord}_\infty(g) : g \in \mathbb{C}[F_1, \dots, F_{m-1}]$. STEP 2 now takes care of products of the F_i.

STEP 2: To start with STEP 2, take the output $L := \{F_1, \dots, F_{m-1}\}$ from STEP 1. To do an "elementary step", we now consider $L^* := L \cup \{F_i F_j\}$ with fixed i, j such that $1 \le i \le j \le m-1$, and apply STEP 1 to L^* to obtain $\{G_1, \dots, G_{m-1}\} \subseteq M^\infty(N)$. Then these G_i have the property that

$$\mathbb{C}[t, F_1, \dots, F_{m-1}] = \mathbb{C}[t, F_1, \dots, F_{m-1}, F_i F_j] = \mathbb{C}[t, G_1, \dots, G_{m-1}]$$

and

$$\{-\operatorname{ord}_\infty(G_1), \dots, -\operatorname{ord}_\infty(G_{m-1})\} \equiv \{1, \dots, m-1\} \pmod m.$$

Moreover, one can easily check that either $\{G_1, \dots, G_{m-1}\} = \{F_1, \dots, F_{m-1}\}$ or

$$-\sum_{i=1}^{m-1} \operatorname{ord}_\infty(G_i) < -\sum_{i=1}^{m-1} \operatorname{ord}_\infty(F_i). \tag{38}$$

The next "elementary step" is made with choosing $G_i, G_j \in \{G_1, \dots, G_{m-1}\}$ such that $F_i F_j \ne G_i G_j$ and applying STEP 1 to $L^* := \{G_1, \dots, G_{m-1}\} \cup \{G_i G_j\}$. Such "elementary steps" are continued until the set $\{G_1, \dots, G_{m-1}\}$ does not change anymore, and the Algorithm MODULE GENERATORS returns the output

$$(g_0, g_1, \dots, g_{m-1}) := (1, G_1, \dots, G_{m-1}).$$

Termination is guaranteed by (38).

Note. There are more efficient, but less transparent strategies for this algorithm.

8. The Ramanujan-Kolberg Algorithm

To describe the steps of Radu's Ramanujan-Kolberg Algorithm [23] (in short called "RK Algorithm" below) we need to specify certain sets of eta quotients:

$$E(N) := \Big\{ \prod_{\delta|N} \eta(\delta\tau)^{r_\delta} : (r_\delta)_{\delta|N} \text{ is proper} \Big\},$$

where an integer sequence $(r_\delta)_{\delta|N}$ is called proper if

$$\sum_{\delta|N} r_\delta = 0, \ \sum_{\delta|N} \delta r_\delta \equiv \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \text{ and } \prod_{\delta|N} \delta^{|r_\delta|/2} \in \mathbb{N}.$$

Using transformation properties of the η-function one sees that $E(N) \subseteq M(N)$. The set $E(N)$ and also

$$E^\infty(N) := E(N) \cap M^\infty(N)$$

clearly form a (multiplicative) monoid. The following, algorithmically important lemma traces back to Newman; see [23].

Lemma 8.1. *For each $N \in \mathbb{N}^*$ there exists $\mu \in E^\infty(N)$ such that*

$$\mathrm{ord}_{a/c}(\mu) > 0 \text{ for all } \frac{a}{c} \in \mathbb{Q}.$$

As a consequence, multiplying functions $f \in M(N)$ with suitable powers μ^k of such eta quotients μ, one can remove all the poles sitting at the cusps $\left[\frac{a}{c}\right]$ whenever $\frac{a}{c} \in \mathbb{Q}$; i.e., resulting in $\mu^k f \in M^\infty(N)$.

RK Algorithm Step 1. For the INPUT as specified in Section 4, compute $\sigma \in \mathbb{Q}$ and an integer sequence $(s_\delta)_{\delta|N}$ together with the finite set $P_{m,r}(\ell)$ of integers such that

$$q^\sigma \prod_{\delta|N} \prod_{n=1}^{\infty} (1-q^{\delta n})^{s_\delta} \prod_{\ell' \in P_{m,r}(\ell)} \sum_{n=0}^{\infty} a_r(mn+\ell')q^n \in M^\infty(N). \tag{39}$$

In [23] one finds conditions for the existence of such σ and (s_δ) together with algorithms to compute them.

Example. In the case of (16) the algorithm determines

$$q^{\frac{17}{24}} \frac{\eta(\tau)^8}{\eta(7\tau)^7} \sum_{n=0}^{\infty} p(7n+5)q^n \in M^\infty(7). \tag{40}$$

Define

$$\langle E(N)\rangle_{\mathbb{C}} := \text{the } \mathbb{C}\text{-vector space generated by the elements of } E(N).$$

Note. This vector space is infinite dimensional and it contains the constant functions; i.e., $1 \in \langle E(N)\rangle_{\mathbb{C}}$.

Suppose $f \in M^\infty(N)$ is given in a form like on the left side of (39). The goal of the RK Algorithm is to find $e_1, \ldots, e_k \in E(N)$ such that for $c_1, \ldots, c_k \in \mathbb{C}$:

$$f = c_1 e_1 + \cdots + c_k e_k \in \langle E(N)\rangle_{\mathbb{C}} \cap M^\infty(N). \tag{41}$$

Suppose that

$$\langle E(N)\rangle_{\mathbb{C}} \cap M^{\infty}(N) = \langle E^{\infty}(N)\rangle_{\mathbb{C}} \tag{42}$$

where

$$\langle E^{\infty}(N)\rangle_{\mathbb{C}} := \text{the } \mathbb{C}\text{-vector space generated by the elements of } E^{\infty}(N).$$

Then the $e_1, \ldots, e_k$ in (41) can be chosen from $E^{\infty}(N)$. Owing to the next lemma, and to Theorem 8.3 below, there is a finite presentation of the infinite vector space $\langle E^{\infty}(N)\rangle_{\mathbb{C}}$.

Lemma 8.2. *The (multiplicative) monoid $E^{\infty}(N)$ is finitely generated.*

RK Algorithm Step 2. Compute generators $E_1, \ldots, E_m$ of $E^{\infty}(N)$ according to [23, Lemma 25].

Note. The generators can be computed using algorithms that solve Diophantine problems like (3).

Example. The RK Algorithm tells that the monoid $E^{\infty}(7)$ is generated by $E_1 := \left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^4 \in E^{\infty}(7)$. Consequently,

$$\begin{aligned}\langle E^{\infty}(7)\rangle_{\mathbb{C}} =& \{c_1 e_1 + \cdots + c_j e_j : j \in \mathbb{N}, c_i \in \mathbb{C},\ e_i \in E^{\infty}(7)\}\\ =& \mathbb{C}[E_1] = \text{polynomials in } E_1 \text{ with coefficients in } \mathbb{C}.\end{aligned}$$

As an alternative to a polynomial ring presentation, $\langle E^{\infty}(7)\rangle_{\mathbb{C}}$ can be also viewed as $\mathbb{C}[E_1]$-module over the trivial generator $1 \in E^{\infty}(7)$; in notation:

$$\langle E^{\infty}(7)\rangle_{\mathbb{C}} = \langle 1\rangle_{\mathbb{C}[E_1]}.$$

Indeed, this is possible in general by the following theorem extracted from [23]:

Theorem 8.3. *Suppose the multiplicative monoid $E^{\infty}(N)$ is generated by $E_1, \ldots, E_m$. Then there are elements t and $z_1, \ldots, z_n \in \langle E^{\infty}(N)\rangle_{\mathbb{C}}$ such that the $\mathbb{C}$-vector space $\langle E^{\infty}(N)\rangle_{\mathbb{C}} = \mathbb{C}[E_1, \ldots, E_m]$ can be represented as a $\mathbb{C}[t]$-module freely generated by $z_1, \ldots, z_n$; i.e.,*

$$\langle E^{\infty}(N)\rangle_{\mathbb{C}} = \langle 1, z_1, \ldots, z_n\rangle_{\mathbb{C}[t]}.$$

RK Algorithm Step 3. Apply Algorithm MODULAR GENERATORS from Section 7 to $\{E_1, \ldots, E_m\}$.

Note. All these computations rely on the fact that we are dealing with modular functions in $M^{\infty}(N)$. As already mentioned, for such functions it is sufficient to handle the principal part of the Laurent series expansions in powers of $q = e^{2\pi i\tau}$ at $\frac{a}{c} = \infty$. Also the last step benefits from this fact.

RK Algorithm Step 4. Let $f(\tau)$ denote the left hand side of (39) and let

$$\langle E^{\infty}(N)\rangle_{\mathbb{C}} = \langle 1, z_1, \ldots, z_n\rangle_{\mathbb{C}[t]}$$

be the output of Step 3. Following the steps as described in the proof of Theorem 7.1, compute polynomials $p_0(t), \ldots, p_n(t) \in \mathbb{C}[t]$ such that

$$f = p_0(t) + p_1(t)z_1 + \cdots + p_n(t)z_n.$$

Example. Define

$$\sum_{k=0}^{\infty} L(k)q^k := \prod_{j=1}^{\infty} \frac{1}{1-q^{2j-1}} = \prod_{j=1}^{\infty} \frac{1-q^{2j}}{1-q^j}.$$

Note. This is the generating function of partitions into odd parts; we started with the Lecture Hall partition problem (2) where such partitions arise but with a restriction on the size of the parts.

RK Step 1 with $M = 2$, $r = (r_1, r_2) = (-1, 1)$, $m = 7$, and $\ell = 3$ delivers $P_{m,r}(3) = \{3, 4, 6\}$ and

$$q^{-5}\prod_{j=1}^{\infty}\frac{(1-q^j)^{13}(1-q^{7j})^8}{(1-q^{2j})^5(1-q^{14j})^{16}} \times \sum_{n=0}^{\infty}L(7n+3)q^n\cdot\sum_{n=0}^{\infty}L(7n+4)q^n\cdot\sum_{n=0}^{\infty}L(7n+6)q^n \in M^{\infty}(14). \tag{43}$$

RK Step 2 gives that the multiplicative monoid $E^{\infty}(14)$ is generated by

$$E_1 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^1\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^7\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-1} \in E^{\infty}(14),$$

$$E_2 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^8\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-8} \in E^{\infty}(14),$$

$$E_3 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-5}\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^5\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-13} \in E^{\infty}(14),$$

$$E_4 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^1\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^3\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-7} \in E^{\infty}(14),$$

$$E_5 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^5\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^7\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-11} \in E^{\infty}(14),$$

and

$$E_6 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-2}\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^6\left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-10} \in E^{\infty}(14).$$

RK Step 3 computes that

$$\langle E^{\infty}(14)\rangle_{\mathbb{C}} = \langle 1, E_4\rangle_{\mathbb{C}[E_1]};$$

i.e., $t = E_1$, $z_1 = E_4$. Denoting the function in (43) by f, RK Step 4 outputs

$$f = 8(p_0(t) + p_1(t)z_1)$$

where $p_0(t) = -16t + 9t^2$ and $p_1(t) = 2t$.

Note. It is easily checked that this relation implies for all $n \in \mathbb{N}$:

$$L(7n+3) \equiv L(7n+4) \equiv L(7n+6) \equiv 0 \pmod 2. \tag{44}$$

In a different setting, Gordon and Ono [11] have shown the strikingly general result that almost all values of $L(n)$ are divisible by 2^k for any $k \in \mathbb{N}$.

Radu's "Ramanujan-Kolberg" package also delivers that

$$\sum_{n=0}^{\infty}L(7n)q^n\cdot\sum_{n=0}^{\infty}L(7n+1)q^n\cdot\sum_{n=0}^{\infty}L(7n+5)q^n$$
$$=q^6\prod_{j=1}^{\infty}\frac{(1-q^{2j})^5(1-q^{14j})^{16}}{(1-q^j)^{13}(1-q^{7j})^8}(3E_1^3 + 24E_1^2 + 64E_1)$$

and

$$\sum_{n=0}^{\infty}L(7n+2)q^n = q^3\prod_{n=1}^{\infty}\frac{(1-q^{14j})^8}{(1-q^j)^3(1-q^{2j})(1-q^{7j})^4}(8E_1 + E_4 - 8).$$

The summands on the right hand sides have no common integer factor, so no divisibility results from these identities. □

In Section 4 we remarked that so far in the literature one could not find any Ramanujan-Kolberg identity presenting a witness for the congruence (13). In the next example we show what the Ramanujan-Kolberg package produces for this case.

Example. RK Step 1 computes that

$$q^{-14}\prod_{j=1}^{\infty}\frac{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}{(1-q^{22j})^{22}} \in M^{\infty}(22).$$

RK Step 2 computes $E_1,\dots,E_8$ as generators of $E^{\infty}(22)$ where

$$E_1(\tau)=\frac{\eta(2\tau)^8\eta(11\tau)^4}{\eta(\tau)^4\eta(22\tau)^8},\ E_2(\tau)=\frac{\eta(2\tau)\eta(11\tau)^{11}}{\eta(\tau)\eta(22\tau)^{11}},\ E_3(\tau)=\frac{\eta(\tau)^7\eta(11\tau)^3}{\eta(2\tau)^3\eta(22\tau)^7},$$

and the other E_i as in [23].

RK Step 3 computes

$$\langle E^{\infty}(22)\rangle_{\mathbb{C}}=\langle 1,z_1,z_2\rangle_{\mathbb{C}[t]}$$

where

$$t=\frac{1}{11}E_1-\frac{1}{8}E_2+\frac{3}{88}E_3,\ z_1=-3+\frac{2}{11}E_1-\frac{1}{8}E_2-\frac{5}{88}E_3,$$
$$z_2=-\frac{3}{11}E_1+\frac{5}{4}E_2+\frac{1}{44}E_3.$$

RK Step 4 computes the following Ramanujan-Kolberg identity witnessing the congruence (13). (Note that 11 divides each of the coefficients on the right hand side.)

$$\begin{aligned}\sum_{n=0}^{\infty}p(11n+6)q^n =&q^{14}\prod_{j=1}^{\infty}\frac{(1-q^{22j})^{22}}{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}\\ &\times\{(1078t^4+13893t^3+31647t^2+11209t-21967)\\ &+z_1(187t^3+5390t^2+594t-9581)\\ &+z_2(11t^3+2761t^2+5368t-6754)\}.\end{aligned}$$

Note. The sufficiently involved structure of this witness identity might explain why Ramanujan (and others) did not come up with such a relation.

Finally we want to come back to the assumption (42) made above. Actually we do not know whether this is true in general. Nevertheless, this "blind spot" is no restriction to the framework in which the RK Algorithm works owing to the following lemma in [23].

Lemma 8.4. *Let $\mu\in E^{\infty}(N)$ be as in Lemma (8.1). Then there exists an $n\in\mathbb{N}$, computable in finitely many steps, such that*

$$\mu^n(\langle E(N)\rangle_{\mathbb{C}}\cap M^{\infty}(N))\subseteq\langle E^{\infty}(N)\rangle_{\mathbb{C}}.$$

If the left side of (39) has a presentation of the type (41), then the Lemma 8.4 tells us that we have to apply a preprocessing step before executing the RK Algorithm. Namely, multiplying the input (39) by μ^n allows us to find the e_j in (41) as elements in $E^{\infty}(N)$ as desired.

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