Spatial Straight Line Linkages by Factorization of Motion Polynomials

Zijia Li
Johan Radon Institute for
Comp. and Appl. Mathematics
Austrian Academy of Sciences
zijia.li@oeaw.ac.at

Josef Schicho
Johan Radon Institute for
Comp. and Appl. Mathematics
Austrian Academy of Sciences
josef.schicho@risc.jku.at

Hans-Peter Schröcker
Unit Geometry and CAD
University Innsbruck
hans-peter.schroecker@uibk.ac.at

ABSTRACT

We use the recently introduced factorization of motion polynomials for constructing overconstrained spatial linkages with a straight-line trajectory. Unlike previous examples of such linkages by other authors, they are single loop linkages and the end-effector motion is not translational. In particular, we obtain a number of linkages with four revolute and two prismatic joints and a remarkable linkage with seven revolute joints one of whose joints performs a Darboux motion.

1 Introduction

Spatial mechanisms with exact straight-line trajectories are rare. The first non-trivial example is due to [1]. It has the property that all trajectories are straight-lines and is nowadays called Sarrus' 6R linkage. Multi-looped linkages, composed of spherical and planar parts, with one straight-line trajectory were presented by Pavlin and Wohlhart in [2]. Other mechanisms with non-trivial straight-line trajectories include the "Wren platform" and some of its variants [3, 4] or the generators for the vertical Darboux motion of Lee and Hervé [5].

In this article we construct new single-looped linkages with a straight-line trajectory. In contrast to Sarrus' linkage, the end-effector motion is not purely translational. In contrast to the examples given by Pavlin and Wohlhart, the linkage is single-looped and in general not composed of planar or spherical parts. In a special case, we show that the Darboux motion can be uniquely decomposed in a rotation and a circular translation and use this for the construction of Darboux linkages which do not contain prismatic or cylindrical joints and, in contrast to [5], perform the general Darboux motion. To define the scope of this paper more precisely: We systematically construct closed-loop straight-line linkages with only revolute or prismatic joints whose coupler motion is neither planar, nor spherical, nor translational and has degree three in the dual quaternion model of rigid body displacements.

We do not claim that spatial straight-line linkages are of particular relevance to engineering sciences. But it should be evident after reading this paper that we gained new insight to some well-known planar and spatial motions. The presented ideas may be extended to other, more useful, synthesis tasks. Our basic tool is factorization of motion polynomials, as introduced in [6]. While that paper presents a general theory and algorithmic treatment for the generic case, a good deal of this paper deals with non-generic cases and thus furthers our understanding of motion polynomial factorization. The basic idea is to decompose a rational end-effector motion, parameterized by a motion polynomial, in different ways into the product of coupled rotations or translations. These rotations/translations give rise to open chains with revolute/prismatic joints that are capable of performing the given end-effector motion. Suitably combining these open chains then yields mechanisms with one degree of freedom whose end-effector follows the prescribed rational motion. Our key-tool is the factorization of motion polynomials into products of linear factors which correspond to the rotations or translations in the decomposition of a rational motion.

2 Preliminaries

We continue with a brief introduction to the dual quaternion model of rigid body displacements. In particular, we derive the straight-line constraint in that model and introduce the notion of "motion polynomials".

2.1 The straight-line constraint

We begin by deriving the constraint equation for all direct isometries of Euclidean three-space that map one point p onto a straight-line L. We do this in terms of dual quaternions, making use of the well-known isomorphism between the group SE(3) of direct isometries and the factor group of unit dual quaternions modulo ± 1. A dual quaternion is an expression of the form $h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} + \varepsilon(h_4 + h_5\mathbf{i} + h_6\mathbf{j} + h_7\mathbf{k})$. Multiplication of dual quaternions is defined by the rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, $\varepsilon^2 = 0$, $\mathbf{i}\varepsilon = \varepsilon\mathbf{i}$, $\mathbf{j}\varepsilon = \varepsilon\mathbf{j}$, $\mathbf{k}\varepsilon = \varepsilon\mathbf{k}$. We denote the set of dual quaternions by $\mathbb{DH}$. The dual quaternion h may be written as $h = x + \varepsilon y$ with ordinary quaternions x, $y \in \mathbb{H}$, the *primal* and *dual part* of h, respectively. The conjugate dual quaternion is defined as $\overline{h} = \overline{x} + \varepsilon\overline{y}$ and ordinary quaternions are conjugated by multiplying the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with -1. Conjugation obeys the rule $\overline{(hk)} = \overline{k}\,\overline{h}$ for any two $h, k \in \mathbb{DH}$. The norm of the dual quaternion h is $\|h\| := h\overline{h} = \|x\| + \varepsilon(x\overline{y} + y\overline{x})$. It can be immediately confirmed that it is a *dual number,* that is, primal and dual part are real numbers.

After projectivizing $\mathbb{DH}$, we obtain Study's kinematic mapping $\mathrm{SE}(3) \to P^7$, see for example [7]. The unit dual quaternion $x + \varepsilon y$ acts on $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ according to

$$1 + \varepsilon(p'_1\mathbf{i} + p'_2\mathbf{j} + p'_3\mathbf{k}) = (x - \varepsilon y)(1 + \varepsilon(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}))(\overline{x + \varepsilon y}). \tag{1}$$

The dual quaternion $x + \varepsilon y$ is projectively equal to a unit norm dual quaternion, if the Study condition $x\overline{y} + y\overline{x} = 0$ is fulfilled and $x \neq 0$. In this case, the action of $x + \varepsilon y$ on p is still defined as in (1) but the right-hand side has to be divided by $x\overline{x}$. It is hence a rational expression in the components of x and y.

Straight-line constraints in the dual quaternion setting are the topic of [8, Section 5.1]. We re-derive a dual quaternion condition for a particular case. Choosing appropriate Cartesian coordinates in the moving frame, we may assume $p = (0,0,0)$. Similarly, it is no loss of generality to assume that $\{(t,0,0) \mid t \in \mathbb{R}\}$ is the set of points on L. Writing $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ and $y = y_0 + \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3$, the second and third coordinate of p' vanish if and only if

$$x_0y_2 - x_1y_3 - x_2y_0 + x_3y_1 = 0, \quad x_0y_3 + x_1y_2 - x_2y_1 - x_3y_0 = 0. \tag{2}$$

This system has to be augmented with the Study condition

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{3}$$

It is straightforward to check that the system of equations (2) and (3) has the solution

$$x \equiv \mathbf{i}y \quad \text{or, equivalently,} \quad y \equiv -\mathbf{i}x \tag{4}$$

where "$\equiv$" denotes equality in projective sense, that is, up to multiplication with constant scalars.

2.2 Motion polynomials

Denote the set of all polynomials in the indeterminate t by $\mathbb{DH}[t]$ and, similarly, by $\mathbb{H}[t]$ and $\mathbb{R}[t]$ the set of polynomials in t with quaternion or real coefficients, respectively. A parameterized rational motion is given by a polynomial $C = X + \varepsilon Y \in \mathbb{DH}[t]$ with X, $Y \in \mathbb{H}[t]$ and the additional properties $X \neq 0$ and $X\overline{Y} + Y\overline{X} = 0$. The latter is equivalent to $C\overline{C} \in \mathbb{R}[t]$ (the conjugate polynomial is obtained by conjugating the coefficients) and is the polynomial form of the Study condition (3). Both conditions ensure that $C(t)$ describes a rigid body displacement for all $t \in \mathbb{R}$ with the exception of possible real zeros of X. By a re-parametrization, we may then assume that the leading coefficient of C is invertible whence a suitable change of coordinates makes C even monic (leading coefficient equals 1). These polynomials have been called *motion polynomials* in [6]. Their coefficients are dual quaternions and do not commute. Therefore, additional conventions for notation, multiplication and evaluation are necessary:

– We always write coefficients to the left of the indeterminate t. This convention is sometimes emphasized by speaking of "left-polynomials" but we just use the term "polynomial".

- Multiplication of polynomials uses the additional rule that the indeterminate t commutes with all coefficients: With $C = \sum_{i=0}^{n} c_i t^i$ and $D = \sum_{i=0}^{m} d_i t^i$ we have $CD = \sum_{i=0}^{n+m} e_i t^i$ where $e_i = \sum_{j=0}^{i} c_j d_{i-j}$ for $i = 0, \ldots, n+m$.
- The value of the polynomial $C = \sum_{i=0}^{n} c_i t^i$ at $h \in \mathbb{DH}$ is defined as $C(h) := \sum_{i=0}^{n} c_i h^i$, that is, it is obtained by substituting h for t in the *expanded form*.

Here is a short example to clarify these conventions. Consider the polynomial $C = (t-k)(t-h)$ with $h, k \in \mathbb{DH}$. Its expanded form reads $C = t^2 - (h+k)t + kh$ (we used commutativity of t and h). The dual quaternion h is a zero of C but k is, in general, not: $C(h) = h^2 - (h+k)h + kh = 0$, $C(k) = k^2 - (h+k)k + kh = hk - kh$. Substituting t by k in the factorized form gives a different value. This is clear since factorized form and expanded form are only equivalent under commutativity assumptions.

Above examples suggest a relation between right factors and zeros of motion polynomials that, in fact, holds true in a more general setting. The following lemma has been stated in proved in [6, Lemma 2].

Lemma 1. *Let $C \in \mathbb{DH}[t]$ and $h \in \mathbb{DH}$. Then $t-h$ is a right factor of C (there exists $Q \in \mathbb{DH}[t]$ such that $C = Q(t-h)$) if and only if $C(h) = 0$.*

In order to apply motion factorization for the construction of straight-line linkages, we need to find a polynomial $C = X + \varepsilon Y \in \mathbb{DH}[t]$ that satisfies (4) identically in t. This already implies that C is a motion polynomial. Our construction of straight-line linkages is largely based on the factorization theorem for motion polynomials [6, Theorem 1]. Consider a monic motion polynomial $C = X + \varepsilon Y \in \mathbb{DH}[t]$ of degree n whose primal part $X \in \mathbb{H}[t]$ has no real factor. Then this theorem states that C can in general be written in $n!$ ways as

$$C = (t-h_1)\cdots(t-h_n) \tag{5}$$

with $h_1, \ldots, h_n \in \mathbb{DH}$ representing rotations. Here, the phrase "in general" refers to possible coinciding of two or more factorizations, for example in the case $C = (t-h)^n$. Note that, in contrast to polynomial factorization over real or complex numbers, the linear factors in (5) do not commute. If the primal part X has real factors, it is possible that some of the h_i represent translations but also cases with no or infinitely many factorizations exist. Examples of these situation can be found in Theorem 4 in the appendix.

The algorithm for computing factorizations in generic cases is explained in [6] and, in more algorithmic form, in [9]. A basic understanding of this algorithm is necessary for reading this paper. Therefore, we provide an informal description. A more formal algorithmic description in pseudo-code is given in [9], actual implementations can be found in the supplementary material of [6].

The norm polynomial $C\overline{C}$ is real and non-negative. Hence, it factors into the product $C\overline{C} = M_1 \cdots M_n$ of n quadratic factors such that each factor M_i is either irreducible over $\mathbb{R}$ or the square of a linear factor. In order to compute a factorization of the form (5), we pick one of the quadratic factors, say M_i, and right-divide C by M_i. That is, we compute $Q, R \in \mathbb{DH}[t]$ such that $\deg R \leq 1$ and $C = QM_i + R$. In general, R has a unique zero – the rotation or translation polynomial h_n. Once the rightmost factor h_n has been computed, we compute C_1 such that $C = C_1(t-h_n)$ and repeat above steps with C_1 instead of C. Note that

$$C_1\overline{C_1} = \prod_{j \neq i} M_j$$

such that all factors of the original norm polynomial $C\overline{C}$ will be used during this process. In this sense, we can say that a factor $t - h_i$ or the rotation/translation quaternion h_i itself "corresponds" to one of the quadratic polynomials M_j. Different factorizations come from permutations of these polynomials.

In exceptional cases, the leading coefficient of the linear remainder polynomial R fails to be invertible. Then, the above algorithm will not produce a valid factorization. This does, however, not mean that no factorization exists. In fact, in this paper we will encounter situations with no or infinitely many factorizations of the form (5).

The kinematic interpretation of motion polynomial factorization is that the motion polynomial parameterizes the rational end-effector motion of, in general, $n!$ open chains consisting of n revolute or prismatic joints. Linkages are obtained by suitably combining a sufficient number of these open chains. In case of $\deg C \leq 3$, two suitably chosen open chains are in general sufficient and will result in an overconstrained, single-looped linkage. Constructions of this type are the topic of this paper's main section.

3 Mechanism synthesis

The most general polynomial solution of (4) is given by

$$C = X + \varepsilon Y \quad \text{with} \quad X = \xi P; \quad Y = -\eta \mathbf{i} P; \quad P \in \mathbb{H}[t]; \quad \xi,\ \eta \in \mathbb{R}[t],$$

and P has no real factor. Let us verify that the trajectory of $p = (0,0,0)$ is really a straight line. According to (1), the image p' of p can be read off from

$$1 + p' \equiv (X - \varepsilon Y)(\overline{X + \varepsilon Y}) = (\xi P + \varepsilon\eta\mathbf{i}P)(\overline{\xi P - \varepsilon\eta\mathbf{i}P}) = (\xi P + \varepsilon\eta\mathbf{i}P)(\xi\overline{P} + \varepsilon\eta\overline{P}\mathbf{i}) = P\overline{P}(\xi^2 + 2\varepsilon\xi\eta\mathbf{i}). \tag{6}$$

Indeed, the right-hand side of (6) leads to a point on the line L. More precisely, a parameterized equation of the trajectory is $p'(t) = \frac{2\eta}{\xi}\mathbf{i}$. From this, we conclude that $\eta = 0$ or constant ξ and η yield a constant point p'. The resulting motion is spherical and shall be excluded from further consideration. That is, we can assume $\eta \neq 0$ and ξ, η are not both constant. This implies $\deg P < \deg C$. In order to narrow the focus of this paper, we also wish to avoid $\deg P = 0$ or, more generally, $P \in \mathbb{R}[t]$. This leads to a translation in constant direction — a motion which is planar in multiple ways.[1] By a change of coordinates we can achieve that C is monic whence $\deg\eta < \deg\xi$. Finally, we may transfer constant real factors between P and ξ, so that we can assume that both, P and ξ are monic. Summarizing these constraints, we have:

$$0 \le \deg\eta < \deg\xi, \quad 1 \le \deg P < \deg C \le 3, \quad P \notin \mathbb{R}[t], \quad \xi, P \text{ are monic.}$$

Hence, we only have to discuss two major cases, $\deg P = 1$ and $\deg P = 2$. The former has three sub-cases ($\deg\xi = 1$ and $\deg\eta = 0$, $\deg\xi = 2$ and $\deg\eta = 0$, $\deg\xi = 2$ and $\deg\eta = 1$), the latter only one ($\deg\xi = 1$, $\deg\eta = 0$).

3.1 The case of degree two

We consider the case $\deg P = 2$, $\deg\xi = 1$, and $\deg\eta = 0$ first. The norm polynomial admits the factorization $C\overline{C} = M_1M_2M_3$ where $\xi^2 = M_1$ and $P\overline{P} = M_2M_3$. This is already a special case as one factor, M_1, is not strictly positive. The following theorem gives a relation between the factors of a motion polynomial and the factors of its norm polynomial for this case.

Theorem 1. *The norm polynomial of a motion polynomial factors as $C\overline{C} = \prod_{i=1}^{n} M_i$ with non-negative factors $M_1, \ldots, M_n$ which are either irreducible over $\mathbb{R}$ or the squares of linear polynomials in $\mathbb{R}[t]$. If M is such a square, the corresponding factor $t - h$ in every factorization of C describes a translation.*

The first part of this proposition is already due to [6]. The statement on the translation can also be found there but it is only motivated, not proved.

Proof of Theorem 1. If $t - h$ is a factor corresponding to M, the dual quaternion h is necessarily a common zero of C and M ([6, Lemma 3]). In particular, if $M = (t-r)^2$ with $r \in \mathbb{R}$, we can evaluate the condition $h^2 - 2hr + r^2 = 0$. By [10, Theorem 2.3], this equation can only be satisfied by dual quaternions of primal part $r \in \mathbb{R}$. Hence, h is necessarily a translation quaternion. □

By Theorem 1, every factorization of C contains at least one prismatic joint, corresponding to M_1. Two of them are obtained from the two factorizations,

$$P = (t - h_1)(t - h_2) = (t - h_1')(t - h_2') \quad \text{with} \quad h_1, h_2, h_1', h_2' \in \mathbb{H} \tag{7}$$

of P over $\mathbb{H}$.[2] They are

$$\begin{aligned} C &= (\xi - \varepsilon\eta\mathbf{i})(t - h_1)(t - h_2) && (A)\\ &= (\xi - \varepsilon\eta\mathbf{i})(t - h_1')(t - h_2'). && (A') \end{aligned}$$

The open chains to each factorization consist of two revolute joints, intersecting in the origin p, and one prismatic joint in direction of $\mathbf{i}$. The trajectory of p is trivially a straight line.

Two further factorizations are of the form

$$\begin{aligned} C &= (t - r_1)(t - r_2)(t - s_1) && (B)\\ &= (t - r_1')(t - r_2')(t - s_1) && (B') \end{aligned}$$

[1] Note however, that the factorization of a translation in constant direction does not necessarily lead to planar linkages. An example of this are Sarrus linkages with rational coupler motion.

[2] Recall that P has no real factor and hence always admits a finite number of factorizations.

with rotation quaternions r_1, r_2, r'_1, $r'_2 \in \mathbb{DH}$ and a translation quaternion $s_1 \in \mathbb{R} + \varepsilon\mathbb{H}$.

Finally, there are two factorizations with factors $t - r_1$, $t - r_2$ on the left and factors $t - h_2$, $t - h'_2$ on the right:

$$C = (t - r_1)(t - s_2)(t - h_2) \tag{C}$$
$$= (t - r'_1)(t - s'_2)(t - h'_2). \tag{C'}$$

Here, the translation quaternions are s_2 and s'_2. In each chain, the last revolute axis (corresponding to the factor on the right) contains the origin p of the moving frame.

Assuming that the two factorizations in (7) are really different, a suitable combination of the factorizations (A)–(C') results in spatial linkages with a straight-line trajectory. We will have a closer look at the manifold relations between the involved joint axes. This will deepen our geometric understanding of these linkage classes and provide us with necessary conditions on the linkage's Denavit-Hartenberg parameters.

To begin with, it must be noted that not every combination of two open chains resulting from the factorizations (A)–(C') is admissible for the construction of overconstrained, single looped linkages with one degree of freedom. In order to avoid "dangling" links, we must not combine two factorizations with the same factor at the beginning or at the end. Hence, we have only four essentially different admissible pairings: A–B, A–C', B–C', C–C'. Non-admissible pairings do not give suitable linkages but information on joint axes. If two factorizations have a common factor at the beginning or the end, the remaining factors can be assembled into a closed linkage with four joints. Consider, for example, the factors (A) and (C). Their closure equation simplifies to

$$1 \equiv (\xi - \varepsilon\eta\mathbf{i})(t - h_1)(t - h_2)(\overline{t - h_2})(\overline{t - s_2})(\overline{t - r_1}) \equiv (\xi - \varepsilon\eta\mathbf{i})(t - h_1)(\overline{t - s_2})(\overline{t - r_1}) = (\xi - \varepsilon\eta\mathbf{i})(t - h_1)(t - \overline{s_2})(t - \overline{r_1}).$$

Hence, the axes of the pair (h_1, r_1), and also that of (h'_1, r'_1), (h_2, r_2), and (h'_2, r'_2), are parallel because they are revolute axes in overconstrained RPRP linkages. A similar argument shows that the axes to r_1, r'_1, r'_2, and r_2 define a Bennett linkage. Finally, the axes to $\varepsilon\mathbf{i}$, h_1, h'_1, h_2, and h'_2 intersect in the point p whose trajectory is a straight line. These observations are responsible for special geometric features of the admissible linkages.

Type A–B: The linkage is of type PRRPRR. The second and third axes intersect. The second and sixth axis and the third and fifth axis are parallel.

Type A–C': In this linkage, three consecutive revolute axes (corresponding to h_1, h_2, h'_2) intersect so that we may view it as PSPR linkage. However, because of (7) we have $(t - h_1)(t - h_2)(\overline{t - h'_2}) \equiv t - h'_1$ and the spherical joint can actually be replaced by a revolute joint. It has to be noted that this replacement cuts away the end effector and, thus, changes the end effector motion. One consequence of this coalescence of the S and R joints are the angle equalities $\sphericalangle(\mathbf{i}, r_1) = \sphericalangle(r_1, s_2)$, $\sphericalangle(\mathbf{i}, r'_1) = \sphericalangle(r'_1, s'_2)$. which are known to hold for the corresponding RPRP linkages. Here, the angle between rotation and translation quaternions is to be understood as angle between their respective axis directions.

Type B–C': This linkage of type RRPRPR contains a Bennett triple of revolute axes (axes one, two and six).

Type C–C': This is an RPRRPR linkages where the third and fourth axes intersect. An example is depicted in Figure 1. The linkage differs from Type A–B in the linkage geometry and in the position of the link with straight-line trajectory.

Remark 1. So far, we constructed 6R/P linkages whose end-effector motion contains the component paramterized by C. In particular, these linkages have at least one degree of freedom. A heuristic argument that they cannot have two or more degrees of freedom is as follows: Because of the presence of two P joints, the spherical motion component is a spherical coupler motion with only one degree of freedom. The two non-parallel P joints cannot change this.

3.2 The cases of degree one

Now we turn to the case $\deg P = 1$ and start our discussion with the sub-case $\deg \xi = 1$. The motion polynomial C is of degree two and it is well-known that its factorizations produce either Bennett linkages or, in limiting cases, an RPRP linkage. The latter occurs here because $C = (\xi - \varepsilon\eta\mathbf{i})P$ clearly is a factorization of C. The second factor, P, describes a rotation about an axis through p, the first factor, $\xi - \eta\mathbf{i}$, describes a translation in direction of $\mathbf{i}$. We omit the possible computation of the second pair of revolute and prismatic joints as this gives us no additional insight. Clearly, every point of either rotation axis and in particular the point $p = (0, 0, 0)$ has a straight-line trajectory.

The remaining cases, $\deg P = 1$, $\deg \xi = 2$, and $\deg \eta = 0$ or $\deg \eta = 1$, can be discussed together. Motion polynomial and norm polynomial are $C = (\xi - \varepsilon\eta\mathbf{i})P$ and $C\overline{C} = \xi^2 P\overline{P}$. We distinguish two sub-cases:

In the first case, the polynomial ξ factors over the reals. Then, by Theorem 1, every closed linkage obtained from factorization of C has four prismatic and two revolute joints. The axes of the revolute joints are necessarily parallel and the

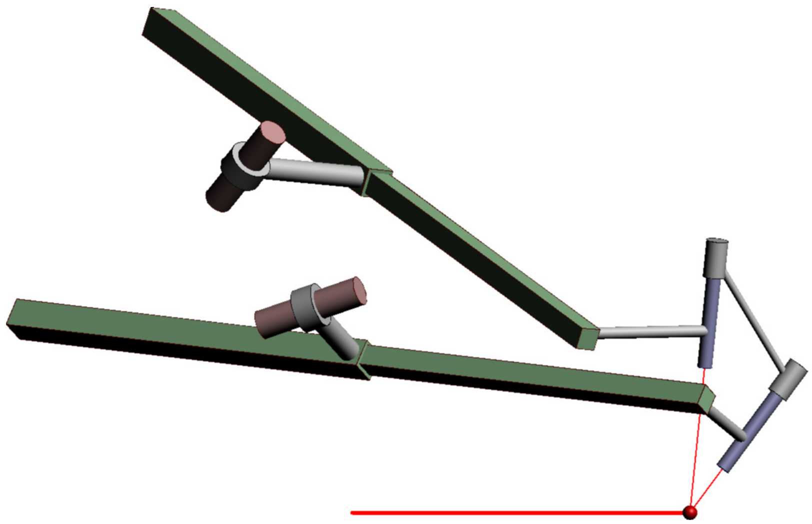

Fig. 1. An RPRRPR linkage with a straight-line trajectory

joint angles for every parameter value t sum to zero. For every fixed revolute joint angle, the linkage admits a one-parametric translational motion along a fixed line. Hence, it has two degrees of freedom and infinitely many straight-line trajectories.

In the second case, the polynomial ξ is *irreducible* over the reals. Then every closed linkage obtained from factorization of C necessarily consists of only revolute joints which makes the envisaged generation of a straight-line trajectory even more interesting. It will turn out that this is only possible under very special circumstances.

Setting $\xi = t^2 + x_1 t + x_0$, $P = t - h$, and $\eta = y_1 t + y_0$ with $h \in \mathbb{H}$ and x_0, x_1, x_2, y_0, $y_1 \in \mathbb{R}$, we assume that C factors as $C = C_1(t-k)$ with a rotation or translation quaternion k. By Lemma 1, k must be a zero of C. We set $k = k_1 + \varepsilon k_2$ with k_1, $k_2 \in \mathbb{H}$ and compute $0 = C(k) = P(k')\xi(k') + \varepsilon\bigl(P(k_1)(k_1k_2 + k_2k_1 + x_1k_2) + k_2\xi(k_1) - \mathbf{i}P(k_1)\eta(k_1)\bigr)$. In order for the primal part to vanish, we have either $P(k_1) = 0$ or $\xi(k_1) = 0$. In the former case, we have $k_1 = h$ and the dual part vanishes only if $k_2 = 0$ or $\xi(k_1) = 0$. If $k_2 = 0$, we have $C_1 = \xi - \mathbf{i}\eta\varepsilon$ and, by Theorem 4 in the appendix, C_1 admits no further factorization. Hence, we can assume $\xi(k_1) = 0$ in any case. This implies $x_1 = -(k_1 + \overline{k_1})$ and $x_0 = k_1\overline{k_1}$.

The quaternion zeros of a quadratic equation are completely described by [10, Thereom 2.3]. Because ξ is irreducible over $\mathbb{R}$ and $\xi(k_1) = 0$, we have

$$k_1 = \frac{1}{2}(-x_1 + w(s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k})) \tag{8}$$

where $w = \sqrt{4x_0 - x_1^2}$ and $s_1^2 + s_2^2 + s_3^2 = 1$. Given k_1 as in (8), the dual part k_2 of k has to satisfy $P(k_1)(k_1k_2 + k_2k_1 + x_1k_2) - \mathbf{i}P(k_1)\eta(k_1) = 0$ and

$k_1\overline{k_2} + k_2\overline{k_1} = 0$. Because of $\overline{k_2} = -k_2$, the second equation implies $k_1k_2 = k_2\overline{k_1}$. We plug this in the first equation and find

$$0 = P(k_1)(k_2\underbrace{(\overline{k_1} + k_1)}_{-x_1} + x_1k_2) - \mathbf{i}P(k_1)\eta(k_1) = -\mathbf{i}P(k_1)\eta(k_1).$$

This is only possible if $P(k_1) = 0$. Hence, we have $k_1 = h$, $x_1 = -h - \overline{h}$ and $x_0 = h\overline{h}$ or, equivalently, $P\overline{P} = \xi$. We will prove in Theorem 2 below that the motion parameterized by C is the well-known Darboux motion, see [5, 11] or [12, Chapter 9, §3]. This is the unique non-planar, non-spherical and non-translational motion with only planar trajectories. It is the composition of a planar elliptic motion and a harmonic oscillation perpendicular to the plane of the elliptic motion. Its trajectories are ellipses with the same major axis length and some trajectories indeed degenerate to straight-line segments.

Theorem 2. *Unless h lies in the linear span of $\mathbf{j}$ and $\mathbf{k}$, the motion parameterized by $C = \xi P - \mathbf{i}\eta\varepsilon P \in \mathbb{DH}[t]$ with $P = t - h \in \mathbb{H}[t] \setminus \mathbb{R}[t]$, $\xi = P\overline{P}$, $\eta \in \mathbb{R}[t]$, $\eta \neq 0$, $\deg \eta \leq 1$ is a* Darboux motion.

Proof. Using $P\overline{P} = \xi$, we compute the parametric equation $\xi^{-1}(2\eta\mathbf{i} + P(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\overline{P}$ for the trajectory of a point (x, y, z). We see that all coordinate functions are at most quadratic. Hence, all trajectories are planar. Since η is different from zero, it is not a spherical motion. Because of our assumptions on h, it is no planar or translational motion. $\square$

We already excluded translational end-effector motions from our considerations and can therefore focus on the factorization and linkage construction for Darboux motions, given by C as in Theorem 2. Algorithmic factorization, as explained in Section 2.2 fails for Darboux motions because of the real factor ξ in the primal part. Thus, a special discussion is necessary. We already saw previously, that right factors are necessarily of the form $t-(h+\varepsilon k_2)$. Conversely, any linear polynomial of that form is really a right factor. The factorization is $C = C_1(t-(h+\varepsilon k_2))$ where

$$C_1 = \xi + \varepsilon D \tag{9}$$

and, with $k_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$,

$$D = ((a-y_1)\mathbf{i} + b\mathbf{j} + c\mathbf{k})t - ah_1 + bh_2 + h_3c - (h_0a + h_2c - h_3b + y_0)\mathbf{i} - (h_0b - h_1c + h_3a)\mathbf{j} - (h_0c + h_1b - h_2a)\mathbf{k}. \tag{10}$$

The factorizability of C_1 is discussed in Theorem 4 in the appendix. Summarizing the results there, we can say the following:

- The motion parameterized by C_1 is a planar translational motion whose trajectories are rational of degree two (or less).
- It admits factorizations if and only if it parameterizes a circular translation. In this exceptional case, the assumptions of [6, Theorem 2] are not met and the motion polynomial admits infinitely many factorizations, corresponding to the multiple generation of a circular translation by parallelogram linkages.
- A criterion for circular translations is $\xi \equiv D\overline{D}$.

Thus, we only have to answer, under which conditions on a, b, c Equation (9) is a circular translation or, equivalently, ξ is a factor of $D\overline{D}$. The latter gives convenient linear equations for a, b, c. Writing $D\overline{D} = z_2t^2 + z_1t + z_0$ where D is as in (10), the linear system to solve is

$$z_0x_1 - z_1x_0 = z_0x_2 - z_2x_0 = z_1x_2 - z_2x_1 = h_1a + h_2b + h_3c = 0. \tag{11}$$

This overconstrained system has a matrix M. The greatest common divisor of all 3×3 minors of M is $\Delta := 4(h_2^2+h_3^2)((h_0y_1+y_0)^2 + y_1^2(h_1^2+h_2^2+h_3^2))$. Again, we need to distinguish two cases:

If $h_2 = h_3 = 0$, the motion is the composition of a rotation about $\mathbf{i}$ and a translation in direction $\mathbf{i}$, that is, a *vertical Darboux motion.* Because P is not a real polynomial, h_1 is different from zero and we necessarily have $a = 0$. This leaves us with three conditions on the solubility: $y_1(h_0y_1+y_0) = y_0((h_0^2+h_1^2)y_1 + h_0y_0) = (h_0^2+h_1^2)y_1 + y_0^2 = 0$. A straightforward discussion shows that either h_1 or y_1 vanish. But both, $h_1 = 0$ and $y_1 = 0$ have been excluded previously. Hence, the vertical Darboux motion allows no factorizations into the product of three linear factors.

If h_2 and h_2 are not both zero, Δ cannot vanish and the system (11) has the unique solution

$$a = \frac{y_1}{2}, \quad b = \frac{y_0h_3 + y_1(h_0h_3 - h_1h_2)}{2(h_2^2+h_3^2)}, \quad c = \frac{y_0h_2 + y_1(h_0h_2 + h_1h_3)}{2(h_2^2+h_3^2)}.$$

In other words, there is precisely one admissible choice for k_2 such that (9) is a circular translation. By an earlier remark, it admits infinitely many factorizations (see also Footnote 3). Thus, we have proved

Theorem 3. *A non-vertical Darboux motion, parameterized by C as in Theorem 2, admits infinitely many factorization into linear motion polynomials. The first two factors on the left describe the same circular translation, the right factor is the same for all factorizations.*

Closed loop linkages for the generation of vertical Darboux motions are described in [5]. Here, it seems that we closely missed the possibility to construct a closed loop linkage with one degree of freedom and only revolute joints that generates a general (non-vertical) Darboux motion. Though we managed to factor the non-vertical Darboux motion in infinitely many ways, we may not form a linkage with one degree of freedom from two factorizations as they have the right factor in common. Nonetheless, there is a way out of this. It requires a "multiplication trick" which will be investigated in more detail and generality in a forthcoming publication. Here, we present the basic idea through an example.

We consider the Darboux motion $C = \xi P - \mathbf{i}\eta\varepsilon P \in \mathbb{DH}[t]$ with

$$\xi = t^2+1, \quad \eta = \frac{5}{2}t - \frac{3}{4}, \quad P = t - h \quad \text{and} \quad h = \frac{7}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}.$$

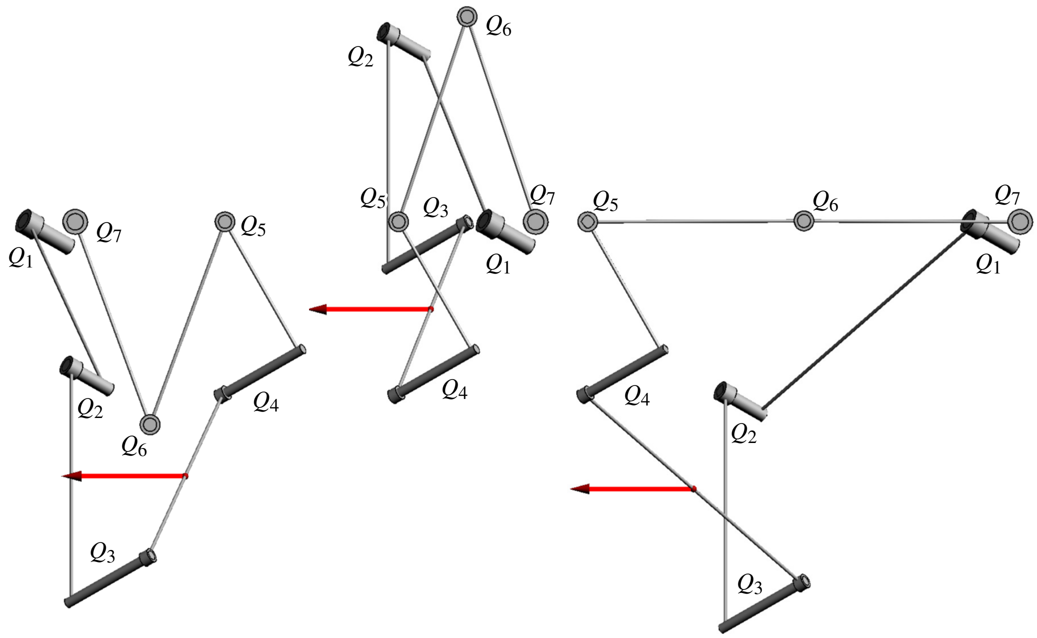

Fig. 2. A 7R linkage that generates a non-vertical Darboux motion.

As seen above, this give us a first factorization $C = Q_1Q_2Q_3$, where

$$Q_1 = t - \frac{7}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} - \frac{5}{4}\varepsilon\mathbf{i} + \frac{43}{64}\varepsilon\mathbf{j} - \frac{97}{64}\varepsilon\mathbf{k}, \quad Q_2 = t + \frac{7}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}, \quad Q_3 = t - \frac{7}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} - \frac{4}{9}\mathbf{k} - \frac{5}{4}\varepsilon\mathbf{i} - \frac{43}{64}\varepsilon\mathbf{j} + \frac{97}{64}\varepsilon\mathbf{k}.$$

In order to obtain a second factorization, we first set the right factor to $Q_4 := P$ and compute C_1 such that $C = C_1Q_4$: $C_1 = t^2 + 1 - \varepsilon\mathbf{i}(\frac{5}{2}t - \frac{3}{4})$. The motion polynomial C_1 parameterizes a translation in constant direction. According to Theorem 4 in the appendix, it cannot be written as the product of two linear motion polynomials. However, after multiplying C_1 by t^2+1, it actually has infinitely many factorizations into products of *three* motion polynomials, one of them being $C'(t^2+1) = Q_7Q_6^2Q_5$, where

$$Q_7 = t - \mathbf{j} - \frac{5}{4}\varepsilon\mathbf{i} - \frac{3}{8}\varepsilon\mathbf{k}, \quad Q_6 = t + \mathbf{j}, \quad Q_5 = t - \mathbf{j} - \frac{5}{4}\varepsilon\mathbf{i} + \frac{3}{8}\varepsilon\mathbf{k}.$$

The multiplicity of the middle factor Q_6 is no coincidence but inherent in the structure of the factorization problem at hand. The kinematic structure to this factorization is an open 4R chain with coinciding second and third axis, that is, actually just a 3R chain. Because $C = Q_1Q_2Q_3$ and $\xi C = Q_7Q_6^2Q_5Q_4$ are projectively equal, we can combine these two factorizations to form a 7R linkage where each rotation is defined by Q_i, $i = 1,\ldots,7$. It can be seen that the axes of Q_1, Q_2 are parallel, as are the axes of Q_3, Q_4 and Q_5, Q_6, Q_7. Moreover, all joint angles are the same – a property that has not yet been observed in non-trivial linkages obtained from motion polynomial factorization.

To complete the above construction, we should check that the configuration space of the 7R linkage is really just a curve. A Gröbner basis computation reveals that this is indeed the case. Note that the configuration curve contains several components, also components of higher genus. One component corresponds to the rational curve parameterized by C. Thus, we have indeed constructed a 7R linkage whose coupler motion is a non-vertical Darboux motion. In Figure 2, we present three configurations of this linkage in an orthographic projection parallel to $\mathbf{j}$. We can observe the parallelism of axes and constancy of one direction during the coupler motion.

4 Conclusions and future research

We have studied spatial straight-line linkages obtained by factorizing a cubic motion polynomial. The mobility and straight-line property of some of the resulting linkages can be explained geometrically while for others the explanation remains algebraic. In the course of this investigation, we showed that a Darboux motion can be decomposed into a circular translation and a rotation and we presented one particular example of a 7R Darboux linkage. A closer investigation of the used "multiplication trick" is left to a forthcoming publication.

Another natural step is to study general trajectory generation in relation to the factorization of motion polynomials. We are already in a position to announce concrete and promising results in this direction.

As already mentioned in the introduction, the engineering relevance of these linkages is probably limited. The present investigation should be rather seen as an exercise in factorization of motion polynomials and a demonstration of what it is capable of. We expect more interesting and applicable linkages to arise from the factorization of motion polynomials in other constraint varieties. Already a cursory glance at the descriptions of constraint varieties in [8] shows that there is plenty of room for further investigations.

A Factorization of quadratic translational motions

In this appendix we prove an auxiliary result that is often referenced in the preceding text. Throughout this section, $C = \xi + \varepsilon D$ is a monic, quadratic motion polynomial with $\xi \in \mathbb{R}[t]$, $\deg \xi = 2$ and $D \in \mathbb{H}[t]$. It is our aim to give a complete description of all possibilities to write C as $C = (t-h)(t-k)$ with rotation or translation quaternions h, $k \in \mathbb{DH}$.

Lets start with some basic properties of the motion C. Because ξ, the primal part of C, is a real polynomial, the motion is translational. Because C is of degree two and monic, the degree of D is at most one. Moreover, $C\overline{C} = \xi(\xi + \varepsilon(D+\overline{D})) \in \mathbb{R}[t]$ implies $\overline{D} = -D$. Conversely, any translational motion of degree two can be written in that way.

The trajectory of the coordinate origin can be parameterized as $x_0^{-1}(x_1, x_2, x_3)$ with polynomials $x_i \in \mathbb{R}[t]$, given by

$$x_0 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{j}) = \xi(\xi - 2\varepsilon(\overline{D} - D)) = \xi(\xi - 2\varepsilon D) \equiv \xi - 2\varepsilon D. \tag{12}$$

We see that this trajectory is rational of degree two at most. Hence, the motion under investigation is a *planar, curvilinear translation.*

Theorem 4. *Let $C = \xi + \varepsilon D$ be a monic, quadratic motion polynomial with* irreducible $\xi \in \mathbb{R}[t]$, $\deg \xi = 2$, $D \in \mathbb{H}[t]$. *Then the following statements are equivalent:*

1. *There exist two rotation quaternions h, $k \in \mathbb{DH}$ such that $C = (t-h)(t-k)$.*
2. *There exist infinitely many rotation quaternions h, $k \in \mathbb{DH}$ such that $C = (t-h)(t-k)$.*
3. *The motion polynomial C parameterizes a circular translation.*
4. *The polynomial ξ divides $D\overline{D}$. (This implies $\xi \equiv D\overline{D}$.)*

Proof. $1 \implies 4$: Write $h = h_1 + \varepsilon h_2$, $k = k_1 + \varepsilon k_2$ with rotation quaternions h_1, h_2, k_1, $k_2 \in \mathbb{H}$. By equating the primal part of $(t-h)(t-k)$ with ξ we find $h_1 + k_1 \in \mathbb{R}$ and $h_1 k_1 \in \mathbb{R}$. This is only possible if $k_1 = \overline{h}_1$. Let us write, for simplicity, $p := h_1 = \overline{k_1}$. Then $\xi = t^2 - (p+\overline{p})t + p\overline{p} = (t-p)(t-\overline{p})$.

Because $k = \overline{p} + \varepsilon k_2$ is a rotation quaternion, we have $pk_2 = -\overline{k_2}\overline{p} = k_2\overline{p}$ (Study condition) and hence $(t-p)k_2 = k_2 t - pk_2 = k_2 t - k_2\overline{p} = k_2(t-\overline{p})$. Using this, the dual part of $(t - h_1 - \varepsilon h_2)(t - k_1 - \varepsilon k_2)$ can be written as $D = -(h_2(t-\overline{p}) + (t-p)k_2) = -(h_2+k_2)(t-\overline{p})$. Compute now $D\overline{D} = (h_2+k_2)(t-\overline{p})(t-p)(\overline{h_2}+\overline{k_2}) = \xi q\overline{q}$ with $q = h_2 + k_2$. Thus, ξ is, indeed, a factor of $D\overline{D}$.

$4 \implies 3$: We already know that C describes a curvilinear translation with rational quadratic trajectories given by (12). The trajectory of the coordinate origin (and hence all other trajectories) are circles if its points at infinity lie on the absolute conic of Euclidean geometry. Algebraically this means that $x_0 = \xi$ divides $x_1^2 + x_2^2 + x_3^2 = 4D\overline{D}$. But this is precisely the assumption.

$3 \implies 2$: A circular translation occurs in infinitely many ways as coupler motion of a parallelogram linkage. This linkage is composed of two 2R chains, each corresponding to one of infinitely many factorizations of C.[3]

The trivial final implication ($2 \implies 1$) completes the proof. □

Remark 2. By Theorem 1, translation quaternions cannot occur in the factorization of C if ξ is irreducible. Hence Theorem 4 gives all factorizations in the case of irreducible ξ.

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[3]This can also be verified at hand of a concrete example. The circular translation $C = 1 + t^2 - \varepsilon(\mathbf{i} + \mathbf{j}t)$ allows the factorizations $C = (t - \mathbf{k} - \varepsilon(-a\mathbf{i} + (1-b)\mathbf{j}))(t + \mathbf{k} - \varepsilon(a\mathbf{i} + b\mathbf{j})))$ with $a, b \in \mathbb{R}$.

References

[1] Sarrus, M., 1853. "Note sur la transformation des mouvements rectilignes alternatifs, en mouvements circulaires; et réciproquement". *C. R. Akad. des Sciences*, p. 1038.

[2] Pavlin, G., and Wohlhart, K., 1992. "On straight-line space mechanisms". In Proceedings of the Sixth International Conference on the Theory of Machines and Mechanisms, pp. 241–246.

[3] Kiper, G., and Söylemez, E., 2011. "Modiefied Wren platforms". In *Proceedings of the 13th IFToMM World Congress in Mechanism and Machine Science*.

[4] Wohlhart, K., 1996. "Kinematotropic linkages". In *Recent Advances in Robot Kinematics*, J. Lenarčič and V. Parenti-Casteli, eds. Kluver Akademic Publisher, pp. 359–368.

[5] Lee, C.-C., and Hervé, J. M., 2012. "On the vertical Darboux motion". In *Latest Advances in Robot Kinematics*, J. Lenarcic and M. Husty, eds. Springer.

[6] Hegedüs, G., Schicho, J., and Schröcker, H.-P., 2013. "Factorization of rational curves in the Study quadric and revolute linkages". *Mech. Mach. Theory,* **69**(1), pp. 142–152.

[7] Husty, M., and Schröcker, H.-P., 2012. "Kinematics and algebraic geometry". In *21st Century Kinematics. The 2012 NSF Workshop*, J. M. McCarthy, ed. Springer, pp. 85–123.

[8] Selig, J., 2011. "On the geometry of point-plane constraints on rigid-body displacements". *Acta Appl. Math.,* **116**(2), pp. 133–155.

[9] Hegedüs, G., Schicho, J., and Schröcker, H.-P., 2015. "Four-pose synthesis of angle-symmetric 6R linkages". *J. Mechanisms Robotics,* **7**(4).

[10] Huang, L., and So, W., 2002. "Quadratic formulas for quaternions". *Appl. Math. Lett.,* **15**(15), pp. 533–540.

[11] Krames, J., 1937. "Zur aufrechten Ellipsenbewegung des Raumes (Über symmetrische Schrotungen III)". *Monatsh. Math. Physik,* **46**(1), pp. 38–50.

[12] Bottema, O., and Roth, B., 1990. *Theoretical Kinematics*. Dover Publications.