Sharp Linkages

Zijia Li

Abstract In this paper, we consider a special kind of overconstrained 6R closed linkages which we call sharp linkages. These are linkages with the property that their bond diagram looks like a $\sharp$ sign. We give a construction of this linkage using the bond theory and motion polynomial factorization methods. These two methods are introduced recently in [6, 7]. Another type of 6R linkages is also introduced. To my knowledge, both types of linkages are new.

Key words: Dual quaternions, motion polynomials, factorization, bond theory, overconstrained 6R linkages.

1 Introduction

In kinematics, a closed 6R linkages with mobility one have been considered by many authors (see [1, 3, 4, 6, 11–13]).

In this paper, we mainly focus on closed 6R linkages. More precisely, we consider a very special type of 6R linkages, which we call sharp linkages. Their bond diagrams look like a $\sharp$ sign. Namely, this bond diagram has two Bennett conditons as the bond diagram of Waldrons double Bennett hybrid, Dietmaier 6R linkages and Bricard plane symmetric 6R linkages [3, Section 4.8.3]. But it is not a special case of those 6R linkages. We also get another 6R linkage which has quasi-symmetric bond diagram. One can find a special angle symmetric 6R linkage in [8] with the same bond diagram. But this new linkage is not angle symmetric, because there is one pair of opposite angles which are not equal.

Our main tools are the bond theory and the factorization of a motion polynomial. These two are based on dual quaternions. In the paper [6], the authors found

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a new 6R linkage by using the factorization of a cubic motion polynomial. We find a quartic motion polynomial in this paper. This quartic motion polynomial has two factorizations which generate two 3R open chains. Using these two chains we can construct the sharp linkage. The main difficulty is to find such quartic motion polynomials. In the future, we want to find all quartic motion polynomial that lead to closed 6R linkages.

The remaining part of the paper is set up as follows. In Section 2, we give the preliminaries we need i.e. dual quaternions, the factorization of a motion polynomial, the bond theory. Section 3 introduces our motivation. Section 4 contains the main result and examples.

2 Preliminaries

In this paper, we mainly use two tools (Bond theory and motion polynomial factorization). Before introducing each of them, let us recall the dual quaternions.

2.1 Dual quaternions

The algebra $\mathbb{DH}$ of dual quaternions is the 8-dimensional real vector space generated by $1, \varepsilon, \mathbf{i}, \mathbf{j}, \mathbf{k}, \varepsilon\mathbf{i}, \varepsilon\mathbf{j}, \varepsilon\mathbf{k}$ (see [6]). Following [6], we can represent a rotation by a dual quaternion of the form $\left(\cot\left(\frac{\phi}{2}\right) - h\right)$, where ϕ is the rotation angle and h is a dual quaternion such that $h^2 = -1$ depending only on the rotation axis. We use projective representations, which means that two dual quaternions represent the same Euclidean displacement if only if one is a real scalar multiple of the other.

The set of all possible motions of a closed 6R linkage is determined by the position of the six rotation axes in some fixed initial configuration. Let L be a 6R linkage given by 6 lines, represented by dual quaternions $h_1, \ldots, h_6$ such that $h_i^2 = -1$ for $i = 1, \ldots, 6$. A configuration (see [6]) is a 6-tuple $(t_1, \ldots, t_6)$, such that the closure condition

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)(t_5 - h_5)(t_6 - h_6) \in \mathbb{R}\backslash\{0\} \tag{1}$$

holds. The configuration parameters t_i – the cotangents of the rotation angles – may be real numbers or ∞, and in the second case we evaluate the expression $(t_i - h_i)$ to 1, the rotation with angle 0. The set of all configurations of L is denoted by K_L. We say L is movable when K_L is a one-dimensional set. Mostly, we will assume, slightly stronger, that there exists an irreducible one-dimensional set for which none of the t_i is fixed. Such a component is called a non-degenerate component. We also exclude the case $\dim_{\mathbb{C}} K_L \geq 2$. Linkages with mobility ≥ 2 do exist, for instance linkages with all axes parallel have mobility 3.

2.2 The factorization of a motion polynomial

In the paper [6], the authors introduced the motion polynomial P which is a monic polynomial in Study quadric of degree n with $P\bar{P} \in \mathbb{R}[t]$. Let $h_1, h_2, \ldots, h_n$ be rotations; using their algorithm, in general, one can compute a factorization

$$P = (t-h_1)(t-h_2)\cdots(t-h_n).$$

One application of the factorization of a motion polynomial is to construct closed linkages by combining the different factorization (which corresponding to different open chains). The difficulty is to find a quartic (or higher degree) motion polynomial which has two factorizations. Furthermore, each of these two factorizations should be corresponding to an open 3R chains. Our main contribution is that we construct such two special quartic motion polynomials. Our construction is based on the bond theory [7].

2.3 The bond theory

Let $L = (h_1, \ldots, h_6)$ be a closed 6R linkage with mobility 1. We assume, for simplicity, that the configuration curve $K_L \subset (\mathbb{P}^1_{\mathbb{R}})^n$ has only one component of dimension 1. Let $K_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$ be the Zariski closure of K_L. We set

$$B := \{(t_1, \ldots, t_n) \in K_{\mathbb{C}} \mid (t_1-h_1)(t_2-h_2)\cdots(t_n-h_n) = 0\}. \tag{2}$$

The set B is a finite set of conjugate complex points on the configuration curve's Zariski closure.

Let β be a bond with coordinates $(t_1, \ldots, t_n)$. By Theorem 2 in [7], there exist indices $i, j \in [n]$, $i < j$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$. If there are exactly two coordinates of β with values $\pm\mathrm{i}$, then we say that β *connects* joints i and j. In general, the situation, is more complicated.

We visualize bonds and their connection numbers by *bond diagrams.* We start with the link diagram, where vertices correspond to links and edges correspond to joints. Then we draw a connecting line between the edges h_i and h_j for each set $\{\beta, \overline{\beta}\}$ of conjugate complex bonds. Multiple connections are possible.

Let us recall [7, Corollary 12] for explaining the connection.

Corollary 1. *For a bond β with $t_i^2 + 1 = t_j^2 + 1 = 0$ and $i < j$, the equality*

$$(t_i - h_i)(t_{i+1} - h_{i+1})\cdots(t_j - h_j) = 0 \tag{3}$$

holds.

In Figure 2, we show some known examples and our new examples with bond diagrams.

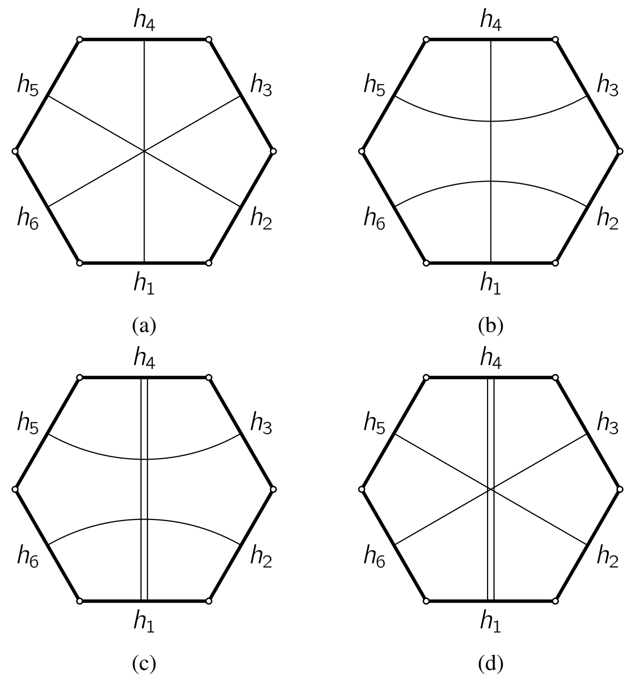

Fig. 2: Bond diagrams for the cube linkage (a), the Waldrons double Bennett hybrid (b), the sharp linkage type one (c), the sharp linkage type two (d)

3 Motivation

In the paper [6], the authors constructed a new 6R linkage by using the factorization of a cubic motion polynomial. It has bond diagram of figure 2(a) which is one of simplest bond diagrams. The other one of simplest bond diagram is figure 2(b) which is known as the Waldrons double Bennett hybrid (see [3] 4.2.5). There is no other 6R linkages with bond diagrams of only three bond connections. Using [7, Theorem 23], one can find the reason as an excise. We consider diagrams with four bond connections, e.g. figure 2(c), (d). There are some other types of bond diagrams with four bond connections. We only consider these two types in this paper.

4 The main results

First, let us make our purpose clear. We want to construct a monic quartic polynomial Q in $\mathbb{DH}[t]$ such that $Q\bar{Q} \in \mathbb{R}[t]$. Furthermore, we can factor Q in two different ways (at least) which both constitute a 3R open chain. Then we can construct a 6R linkage by combining these two factorizations.

Remark 1. Such 6R linkage exists (angle symmetric 6R linkage [8,9]). Up to now, it was not known whether or not there exist such 6R linkages that are not angle symmetric. This paper gives a positive answer.

Now we introduce our procedure for finding such examples.

I. We choose four lines with two different bond connections (3) as following

$$(\mathrm{i}-h_1)(\alpha-h_2)(\beta-h_3)(\mathrm{i}-h_4)=0,$$
$$(\mathrm{i}-h_1)(\alpha'-h_2)(\beta'-h_3)(\mathrm{i}+h_4)=0,$$

where i is the imaginary unit, complex numbers α and β have the same linear relation as α' and β' i.e.

$$\beta=a\alpha+b, \quad \beta'=a\alpha'+b.$$

II. Use these two bond conditions to calculate quartic motion polynomials.
III. Use the factorization algorithm to compute another factorization of the first three factors. This procedure contribute the two lines h_5 and h_6 which we want.
IV. Return the 6R linkage $[h_1,h_2,h_3,h_4,h_5,h_6]$.

Remark 2. There are two options in procedure III (either change the order of second and third or not), which contribute two kinds of 6R linkage with bond diagrams 2(c) and (d).

As the first step is the most important step, we show the details in the following subroutine.

I.a Choose h_2 and h_3 as two random lines with $h_2^2=h_3^2=-1$.
I.b Choose two complex number α and α' where $\alpha\neq\pm\mathrm{i}$ and $\alpha'\neq\pm\mathrm{i}$.
I.c Choose two random real numbers a,b with $a\neq 0$.
I.d Assume that the other two lines have the following formula

$$h_1=(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})+(y_1\mathbf{i}+y_2\mathbf{j}+y_3\mathbf{k})\varepsilon,$$
$$h_4=(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})+(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\varepsilon.$$

I.e Solve the following system for unknowns x_1, x_2, x_3, y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2, v_3

$$\begin{cases} (\mathrm{i}-h_1)(\alpha-h_2)(\beta-h_3)(\mathrm{i}-h_4)=0,\\ (\mathrm{i}-h_1)(\alpha'-h_2)(\beta'-h_3)(\mathrm{i}+h_4)=0,\\ h_1^2=-1, \qquad h_4^2=-1.\end{cases}$$

I.f Choose one solution (all variables are in real) for the next steps.

We add one example to support our procedure. This is a particularly easy example which we found by our procedure.

Input: I.a, I.b, I.c

$$
\begin{aligned}
h_2 &= \left(-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) - \frac{6}{5}\mathbf{k}\varepsilon, \\
h_3 &= \left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) + \left(\frac{76}{49}\mathbf{i} + \frac{24}{49}\mathbf{j} - \frac{30}{49}\mathbf{k}\right)\varepsilon, \\
\alpha &= -\frac{1}{5} - \frac{4}{3}\mathrm{i}, \quad \alpha' = \frac{4}{5} - \frac{1}{2}\mathrm{i}, \\
a &= \frac{5}{2}, \qquad\qquad b = -\frac{3}{4}.
\end{aligned}
$$

Output: Then one can get a numerical solution with 10 digits as following

$$
\begin{aligned}
&x_1 = 0.4058453976, \quad x_2 = -0.9139192147, \quad x_3 = -0.0064173294, \\
&y_1 = 1.244931364, \quad y_2 = 0.5535129673, \quad y_3 = -0.09606363509, \\
&u_1 = -0.6219669897, \quad u_2 = -0.3316117352, \quad u_3 = 0.7093593733, \\
&v_1 = -0.5417103337, \quad v_2 = -1.024569908, \quad v_3 = -0.9539386886.
\end{aligned}
$$

Then the next two steps are for calculating the factorization. We assume that $t_1(t)$ and $t_4(t)$ are quadric rational functions of t, and we also assume that

$$
t_1(\alpha) = \mathrm{i}, \quad t_1(\alpha') = \mathrm{i}, \quad t_4(\alpha) = \mathrm{i}, \quad t_4(\alpha') = -\mathrm{i}. \tag{4}
$$

The quartic motion polynomial is $(t_1(t) - h_1)(t - h_2)(at + b - h_3)$. The other factorization is obtained by multiplying $(t_4(t) - h_4)$ from the right. Then $(t_1(t) - h_1)(t - h_2)(at + b - h_3)(t_4(t) - h_4)$ is a quadric motion polynomial when we remove the real denominators and factors. The next step is to factor this quadric motion polynomial. We show all these details in the following:

Assumption:

$$
\begin{aligned}
t_1(t) &= \frac{t^2 + p_2 t + p_3}{p_4 t + p_5}, \qquad t_4(t) = \frac{t^2 + p'_2 t + p'_3}{p'_4 t + p'_5}, \\
\alpha &= -\frac{1}{5} - \frac{4}{3}\mathrm{i}, \qquad \alpha' = \frac{4}{5} - \frac{1}{2}\mathrm{i}.
\end{aligned}
$$

Do: Solve the linear system (4) for unknowns $p_2, p_3, p_4, p_5, p'_2, p'_3, p'_4, p'_5$.

Output: Then one can get a solution of $t_1(t)$ and $t_4(t)$ as following

$$
t_1(t) = \frac{t^2 - \frac{3}{5}t - \frac{62}{75}}{-\frac{11}{6}t + \frac{29}{30}}, \qquad t_4(t) = \frac{t^2 - \frac{3}{5}t + \frac{38}{75}}{-\frac{5}{6}t + \frac{7}{6}}.
$$

After substituting $t_1(t)$ and $t_4(t)$ into

$$
(t_1(t) - h_1)(t - h_2)(at + b - h_3)(t_4(t) - h_4),
$$

we have a numeric quadric motion polynomial in 10 digits (replacing the real denominators and factors)

$$
\begin{aligned}
t^2 + (-0.3000000000 + 0.6543154994\mathbf{i} - 1.037575959\mathbf{j} + 0.2365105645\mathbf{k} + \\
1.210540727\mathbf{i}\varepsilon - 0.0349528507\mathbf{j}\varepsilon + 0.4738323880\varepsilon\mathbf{k})t - \\
0.2003149450 - 0.0160185109\mathbf{i} + 0.3911798525\mathbf{j} + .2378984092\mathbf{k} - \\
0.9404081633\varepsilon - 1.436504834\mathbf{i}\varepsilon - 0.5526215606\mathbf{j}\varepsilon + 0.0201175896\varepsilon\mathbf{k}.
\end{aligned}
$$

As the norm of this quadric motion polynomial is $(t^2+1)(t^2 - \frac{3}{5}t + \frac{1}{4})$, we can construct two 6R linkages $L_c = [h_1^c, h_2^c, h_3^c, h_4^c, h_5^c, h_6^c]$ and $L_d = [h_1^d, h_2^d, h_3^d, h_4^d, h_5^d, h_6^d]$ (with bond diagram 2(c) and (d)) basing on these two factorization as following (numerically in 10 digits).

$$
\begin{aligned}
h_1^c &= (0.4058453976\mathbf{i} - 0.9139192147\mathbf{j} - 0.0064173294\mathbf{k}) + \\
&\quad (1.244931364\mathbf{i} + 0.5535129673\mathbf{j} - 0.09606363509\mathbf{k})\,\varepsilon, \\
h_2^c &= \left(-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) - \frac{6}{5}\mathbf{k}\varepsilon, \\
h_3^c &= \left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) + \left(\frac{76}{49}\mathbf{i} + \frac{24}{49}\mathbf{j} - \frac{30}{49}\mathbf{k}\right)\varepsilon, \\
h_4^c &= (-0.6219669897\mathbf{i} - 0.3316117352\mathbf{j} + 0.7093593733\mathbf{k}) + \\
&\quad (-0.5417103337\mathbf{i} - 1.024569908\mathbf{j} - 0.9539386883\mathbf{k})\,\varepsilon, \\
h_5^c &= (0.9529670102)\mathbf{i} - 0.2884245020\mathbf{j} - 0.0930869702\mathbf{k}) + \\
&\quad (0.145998817\mathbf{i} - 0.4419436106\mathbf{j} + 2.863982166\varepsilon\mathbf{k})\,\varepsilon, \\
h_6^c &= (0.2731286954)\mathbf{i} - 0.9222061578\mathbf{j} + 0.2737453525\mathbf{k}) + \\
&\quad (1.152141200\mathbf{i} + 0.1418245937\mathbf{j} - 0.6717604788\mathbf{k})\,\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
h_1^d &= h_1^c, \qquad h_2^d = h_2^c, \qquad h_3^d = h_3^c, \qquad h_4^d = h_4^c, \\
h_5^d &= (0.6843121346\mathbf{i} - 0.7290081982\mathbf{j} - 0.0162465108\mathbf{k}) + \\
&\quad (0.7852041130\mathbf{i} + 0.7074301081\mathbf{j} + 1.329661169\mathbf{k})\,\varepsilon, \\
h_6^d &= (-0.0749915882\mathbf{i} - 0.7714194013\mathbf{j} + 0.6318926880\mathbf{k}) + \\
&\quad (1.063341534\mathbf{i} - 1.855957397\mathbf{j} - 2.139571953\mathbf{k})\,\varepsilon.
\end{aligned}
$$

Remark 3. At several places, we used the computer algebra system Maple for more elaborate computations: examples, animations. Because of the length of these computations, it is not reasonable to reproduce them in this paper, but they can be found at our webpage[1]. They can be read with any text editor and verified using Maple 16. One can use a new technique, namely, quad polynomials [10][2], to check mobility

[1] http://people.ricam.oeaw.ac.at/z.li/softwares/sharplinkages.html

[2] http://people.ricam.oeaw.ac.at/z.li/softwares/quadpolynomials.html

from their symbolic Denavit/Hartenberg parameters $[2, 5]^3$ which have complicate square roots.

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[3] One can check that L_c does not fulfill the necessary conditions of Waldrons double Bennett hybrid, Dietmaier 6R linkages or Bricard plane symmetric 6R linkages as an exercise.