The Theory of Bonds II: Closed 6R Linkages with Maximal Genus

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Abstract

In this paper, we study closed linkages with six rotational joints that allow a one-dimensional set of motions. We prove that the genus of the configuration curve of a such a linkage is at most five, and give a complete classification of the linkages with a configuration curve of genus four or five. The classification contains new families.

Keywords: Overconstrained 6R linkage; Dual quaternion; Genus; Quad polynomials

1. Introduction

A linkage is a mechanism composed of a finite number of rigid bodies, called links, and connections between them, called joints. The links move in three-dimensional space, and when two links are connected by a joint, then the relative motion is constrained to a certain subgroup of the group of Euclidean displacements, depending on the type of joint. For instance, a revolute joint ensures that the relative motion is always a rotation around a fixed axis. A linkage consisting of n links that are cyclically connected by n revolute joints is called a closed nR linkage.

In kinematics, one studies the set of all possible configurations of a linkage. If the configuration set has positive dimension, then the linkage is mobile. This is always the case for nR linkages when $n \geq 7$. There are mobile closed nR linkages for $n = 4, 5, 6$. A mobile closed 4R linkage is either planar, or spherical, or a Bennett linkage [1]. For 5R linkages, we have a similar classification that

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has been completed by [2]. The classification of mobile closed 6R linkages is still an open problem.

The theory of bonds was introduced in [3] as a method for the analysis of linkages with revolute joints. The configuration curve of such a linkage can be described by algebraic equations. Intuitively, bonds are points in the configuration curve with complex coefficients where something degenerate happens. For a typical bond of a closed nR linkage, there are exactly two joints with degenerate rotation angles. In this way, the bond “connects” the two links.

The theory of bonds has been used in [3] to give an almost computation-free proof of the classification of closed 5R linkages. The original proof [2] is based on complex computations done with computer algebra. In [4], bonds are used for studying Stewart-Gough platforms with self-motions.

The main result in this paper is the classification of all 6R linkages with a configuration curve of maximal genus (in particular, we assume that the configuration set has dimension 1, which makes it a curve.) In Section 4, we show that the maximum is 5 (examples of genus 5 are well-known, but the fact that 5 is an upper bound is new). In Section 6, we give a classification of all linkages with configuration curve of genus 5 in terms of their Denavit–Hartenberg parameters. It turns out that they come in 4 families, two are well-known and two are new.

In order to derive equations for linkages of the classification, we use a technique which allows to produce polynomials in the Denavit–Hartenberg parameters such that their vanishing is necessary (but in general not sufficient) for the existence of bonds. This technique has been introduced in [5]. For the convenience of the reader, we give in Section 5 an introduction of this technique with sketch of the proofs (for the proof details, we refer to [5]). Section 5 is also logically independent of Section 4, and we hope that some readers are able to solve the equation system for bond diagrams which we could not treat up to now.

It is apparent that the results of bond theory have their main interest in the field of kinematics. However, we also want to address algebraists and geometers, because we hope to serve as an inspiration to use this technique and develop similar ones for solving more questions of interest in kinematics.

2. Preliminary Definitions

In this section we recall several classical concepts and definitions that we need later: linkages and their configuration set and coupler maps, the Study quadric, and dual quaternions.

We denote by SE_3 the group of Euclidean displacements, i.e., the group of maps from $\mathbb{R}^3$ to itself that preserve Euclidean distances and orientation. It is well-known that SE_3 is a semidirect product of the translation subgroup and the orthogonal group SO_3, which may be identified with the stabilizer of a single point.

We denote by $\mathbb{D} := \mathbb{R} + \epsilon\mathbb{R}$ the ring of dual numbers, with multiplication defined by $\epsilon^2 = 0$. The algebra $\mathbb{H}$ is the non-commutative algebra of quaternions, and $\mathbb{DH} := \mathbb{D} \otimes_{\mathbb{R}} \mathbb{H}$. The conjugate dual quaternion $\overline{h}$ of h is obtained by multiplying the vectorial part of h by -1. The dual numbers $\mathrm{N}(h) = h\overline{h}$ and $h + \overline{h}$ are called the *norm* and *trace* of h, respectively.

By projectivizing $\mathbb{DH}$ as a real 8-dimensional vector space, we obtain $\mathbb{P}^7$. The condition that $\mathrm{N}(h)$ is strictly real, i.e. its dual part is zero, is a homogeneous quadratic equation. Its zero set, denoted by S, is called the Study quadric. The linear 3-space represented by all dual quaternions with zero primal part is denoted by E. It is contained in the Study quadric. The complement $S - E$ is closed under multiplication and multiplicative inverse and therefore forms a group, which is isomorphic to SE_3 (see [6, Section 2.4]).

A nonzero dual quaternion represents a rotation if and only if its norm and trace are strictly real and its primal vectorial part is nonzero. It represents a translation if and only if its norm and trace are strictly real and its primal vectorial part is zero. The 1-parameter rotation subgroups with fixed axis and the 1-parameter translation subgroups with fixed direction can be geometrically characterized as the lines on S through the identity element 1. Among them, translations are those lines that meet the exceptional 3-plane E.

Let $n \geq 4$. For the analysis of the configurations of a closed nR linkage with links $o_1, \ldots, o_n$, the actual shape of links is irrelevant; it is enough to know the position of the rotation axes. Exploiting the fact that there is a bijection between lines in $\mathbb{R}^3$ and involutions in SE_3, we describe a closed nR linkage by a sequence $L = (h_1, \ldots, h_n)$ of dual quaternions $h_1, \ldots, h_n$ such that $h_i^2 = -1$ and $h_i \neq \pm h_{i+1}$ for $i = 1, \ldots, n$ (we set $h_{i+kn} = h_i$ and $o_{i+kn} = o_i$ for all $k \in \mathbb{Z}$). The line h_i specifies the joint connecting the links o_{i-1} and o_i. The subgroup of rotations with axis h_i is parametrized by $(t - h_i)_{t \in \mathbb{P}^1}$. The pose of o_i with respect to o_n is then given by a product $(t_1 - h_1)(t_2 - h_2) \cdots (t_i - h_i)$, with $t_1, \ldots, t_i \in \mathbb{P}^1$. Setting $i := n$, we get the closure condition

$$(t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) \in \mathbb{R}^*. \tag{1}$$

The set K of all n-tuples $(t_1, \ldots, t_n) \in (\mathbb{P}^1)^n$ fulfilling (1) is called the *configuration set* of the linkage L.

The dimension of the configuration set is called the *mobility* of the linkage. We are mostly interested in linkages of mobility one. Let $L = (h_1, \ldots, h_n)$ be such a linkage. Let K be its configuration curve. For any pair (o_i, o_j) of links, there is a map

$$f_{i,j} : K \to \mathbb{P}^7, (t_1, \ldots, t_n) \to (t_{i+1} - h_{i+1}) \ldots (t_j - h_j)$$

parametrizing the motion of o_j with respect to o_i. This map is is called coupler map, and the image $C_{i,j}$ is the coupler curve. The *algebraic degree* of the coupler curve is defined as $\deg(C_{i,j}) \deg(f_{i,j})$, where $\deg(C_{i,j})$ is the degree of $C_{i,j}$ as a projective curve, and $\deg(f_{i,j})$ is the degree of $f_{i,j}$ as a rational map $K \to C_{i,j}$.

3. Bonds: Definition and Main Properties

In this section we recall the fundamentals of bond theory, as introduced in [3].

Let $n \geq 4$ be an integer. Let $L = (h_1, \dots, h_n)$ be a closed nR linkage with mobility 1. We assume, for simplicity, that the configuration curve $K \subset (\mathbb{P}^1_{\mathbb{R}})^n$ has only one component of dimension 1 (see Remark 4 for a comment on the reducible case). Let $K_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$ be the Zariski closure of K. We set

$$B := \{(t_1, \dots, t_n) \in K_{\mathbb{C}} \mid (t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) = 0\}. \tag{2}$$

The set B is a finite set of conjugate complex points on the configuration curve's Zariski closure. If K is a nonsingular curve, then we define a bond as a point of B. If K has singularities, then it is necessary to pass to the normalization $N(K)$ of K as a complex algebraic curve, and a bond is then a point on $N(K)$ lying over B. Zero-dimensional components of K never fulfill the equation above and so they have no effect on bonds.

Let β be a bond lying over $(t_1, \dots, t_n)$. By Theorem 2 in [3], there exist indices $i, j \in [n]$, $i < j$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$. If there are exactly two coordinates of β with values $\pm\mathrm{i}$, then we say that β *connects* joints i and j. In general, the situation, is more complicated. Let $\beta \in N(K)$ be a bond; we assume, for simplicity, that it lies over a point $(t_1, \dots, t_n)$ such that no t_i is the infinite point in $\mathbb{P}^1$. For $i, j \in \{1, \dots, n\}$, we define

$$F_{i,j}(\beta) = (t_{i+1}(\beta) - h_{i+1}) \cdots (t_j(\beta) - h_j) \in \mathbb{DH}, \tag{3}$$

The distinction between $F_{i,j}$ and $f_{i,j}$ is necessary because $F_{i,j}$ may vanish at the bonds, and then it does not give a well-defined pose in $\mathbb{P}^7$. We define $v_\tau(i,j)$ as the vanishing order of $F_{i,j}$ at τ. We define the connection number

$$k_\beta(i,j) := v_\beta(i, j-1) + v_\beta(i-1, j) - v_\beta(i,j) - v_\beta(i-1, j-1).$$

We visualize bonds and their connection numbers by *bond diagrams.* We start with the link diagram, where vertices correspond to links and edges correspond to joints. Then we draw $k_\beta(i,j)$ connecting lines between the edges h_i and h_j for each set $\{\beta, \overline{\beta}\}$ of conjugate complex bonds. Since we cannot exclude that $k_\beta(i,j) < 0$, we visualize negative connection numbers by drawing the appropriate number of dashed connecting lines (because the dash resembles a "minus" sign). No linkage in this paper has a negative connection number. Actually, the authors do not know if closed 6R linkages may or may not have bonds with negative connection numbers.

Theorem 1. *The algebraic degree of the coupler curve $C_{i,j}$ can be read off from the bond diagram as follows: Cut the bond diagram at the vertices o_i and o_j to obtain two chains with endpoints o_i and o_j; the algebraic degree of $C_{i,j}$ is the sum of all connections that are drawn between these two components (dashed connections counted negatively).*

Proof. This is a consequence of Theorem 5 in [3]. Note that here we give a different definition of connection numbers, but Lemma 2 in [3] shows that the definitions are equivalent.

The basic idea of the proof is that the algebraic degree of $C_{i,j}$ is $\frac{1}{2}$ times the number of points τ in the configuration curve such that $\mathrm{N}(f_{i,j}(\tau)) = 0$. All these points are bonds, and a closer investigation leads to the statement above. □

Example 1. *We illustrate the procedure for computing the degrees in Figure 1. In order to determine the algebraic degree of the coupler curve $C_{3,5}$, we cut the bond diagram along the line through o_3 and o_5 and count the connections between the two chain graphs. There are precisely two of them, one connecting h_1 with h_4 and one connecting h_2 with h_5. Thus, the algebraic degree $d(3,5)$ of $C_{3,5}$ is two.*

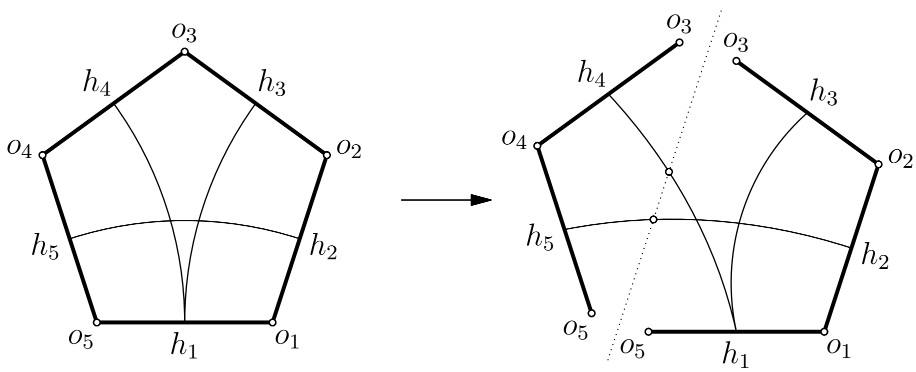

Figure 1: Computing the degree of coupler curves by counting connections in the bond-diagram. There are two connections between the two chains, hence the algebraic degree of the coupler curve $C_{3,5}$ is two.

For a sequence $h_i, h_{i+1}, \ldots, h_j$ of consecutive joints, we define the *coupling space* $L_{i,i+1,\ldots,j}$ as the linear subspace of $\mathbb{R}^8$ generated by all products $h_{k_1} \cdots h_{k_s}$, where $s \geq 0$ and $k_1, \ldots, k_s$ are integers such $i \leq k_1 < \cdots < k_s \leq j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product corresponding to $s = 0$ is included, its value is 1. The *coupling dimension* $l_{i,i+1,\ldots,j}$ is the dimension of $L_{i,i+1,\ldots,j}$ and the *coupling variety* $X_{i,i+1,\ldots,j} \subset \mathbb{P}^7$ is the set of all products $(t_i - h_i) \cdots (t_j - h_j)$ with $t_k \in \mathbb{P}^1$ for $k = i, \ldots, j$ or, more precisely, the set of all equivalence classes of these products in the projective space.

The coupling variety is a subset of the projectivization of the coupling space. The relation between the coupler curve and the coupling variety is described by the "coupler equality" $C_{i,j} = X_{i+1,\ldots,j} \cap X_{i,\ldots,-n+j+1}$.

The relation between bonds and coupling dimensions is described in the following

Theorem 2. *All coupling dimensions $l_{1,\ldots,i}$ with $1 \leq i \leq n$ are even. We have $l_{1,2} = 4$ and $k_\beta(1,2) = 0$ for every bond β. If $k_\beta(1,3) \neq 0$ for some β, then $l_{1,2,3} \leq 6$. If $l_{1,2,3} = 4$, then the lines h_1, h_2, h_3 are parallel or have a common point.*

Proof. This is part of Theorem 1, Theorem 3, and Corollary 3 in [3]. The first statement is a consequence of the fact that the coupling spaces can be given the structure of a complex vector space, because they are closed under multiplication by h_1 from the left. □

We will also use a more precise description of the coupling varieties in each of the three possible cases, which is interesting in itself.

Theorem 3. *If $l_{1,2,3} = 4$, then $X_{1,2,3}$ is a linear projective 3-space, and its parametrization by t_1, t_2, t_3 is a 2:1 map branched along two quadrics in this 3-space.*

If $l_{1,2,3} = 6$, then $X_{1,2,3}$ is a complete intersection of two quadrics in a 5-space and its parametrization by t_1, t_2, t_3 is birational.

If $l_{1,2,3} = 8$, then $X_{1,2,3}$ is a Segre embedding of $(\mathbb{P}^1)^3$ in $\mathbb{P}^7$, and its parametrization by t_1, t_2, t_3 is an isomorphism.

Proof. The first statement is well-known in kinematics. For non-parallel axes it is, for example, implicit in the exposition of [7, Section 5]. Branching occurs for co-planar joint axes. There are two components of the branching surface, and each of the component is the image of a subset of $(\mathbb{P}^1)^3$ in which t_2 is constant.

If $l_{1,2,3} = 6$, then $(\mathrm{i}-h_1)(s_2-h_2)(\pm\mathrm{i}-h_3) = 0$ for some $s_2 \in \mathbb{P}^1_{\mathbb{C}}$, by the proof of Theorem 1 in [3]; we may assume that the third factor is $(+\mathrm{i} - h_3)$. Clearly there is also a complex conjugate relation $(-\mathrm{i} - h_1)(\bar{s_2} - h_2)(-\mathrm{i} - h_3) = 0$. The parametrization $p : (\mathbb{P}^1)^3 \to X_{1,2,3}$ has two base points $(\mathrm{i}, s_2, \mathrm{i})$ and $(-\mathrm{i}, \overline{s_2}, -\mathrm{i})$. We distinguish two cases.

If $s_2 \neq \overline{s_2}$, then we apply projective transformations moving the base points to $(0, 0, 0)$ and (∞, ∞, ∞). The transformed parametrization is

$$(\mathbb{P}^1)^3 \to \mathbb{P}^5, (y_1, y_2, y_3) \mapsto (x_0{:}x_1{:}x_2{:}x_3{:}x_4{:}x_5) = (y_1{:}y_2{:}y_3{:}y_1y_2{:}y_1y_3{:}y_2y_3),$$

which is birational to the quartic three-fold defined by $x_0x_5 = x_1x_4 = x_2x_3$.

If $s_2 = \overline{s_2}$, then we apply projective transformations moving the base points to $(0, \infty, 0)$ and (∞, ∞, ∞). The transformed parametrization is

$$(\mathbb{P}^1)^3 \to \mathbb{P}^5, (y_1, y_2, y_3) \mapsto (x_0{:}x_1{:}x_2{:}x_3{:}x_4{:}x_5) = (1{:}y_1{:}y_3{:}y_1y_2{:}y_1y_3{:}y_2y_3),$$

which is birational to the quartic three-fold defined by $x_0x_4 = x_1x_2$, $x_1x_5 = x_2x_4$.

If $l_{1,2,3} = 8$, then the eight products generating $L_{1,2,3}$ are linearly independent, and it follows that the parametrization is the Segre embedding in the projective coordinate system induced by this basis. □

All bonds connecting h_1 and h_4 satisfy $t_1, t_4 \in \{+\mathrm{i}, -\mathrm{i}\}$. We will prove a lemma that is useful to give an upper bound for the number of bonds in some situations; before that, we need an algebraic lemma.

Lemma 1. *Let $h_1, h_2 \in \mathbb{DH}$ be dual quaternions representing lines (i.e. $h_1^2 = h_2^2 = -1$). Let $\mathbb{D}_{\mathbb{C}} := \mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{DH}_{\mathbb{C}} := \mathbb{DH} \otimes_{\mathbb{R}} \mathbb{C}$ be the extensions of the dual numbers/quaternions to $\mathbb{C}$.*

(a) The left annihilator of $(\mathrm{i} - h_1)$ is equal to the left ideal $\mathbb{DH}_{\mathbb{C}}(\mathrm{i} + h_1)$.

(b) The intersection of this left ideal and the right ideal $(\mathrm{i} - h_2)\mathbb{DH}_{\mathbb{C}}$ is a free $\mathbb{D}_{\mathbb{C}}$-module of rank 1.

(c) The set of all complex dual quaternions x such that $(\mathrm{i} - h_2)x(\mathrm{i} - h_1) = 0$ is a free $\mathbb{D}_{\mathbb{C}}$-module of rank 3.

Proof. For $h_1 = h_2 = \mathbf{i}$, the proofs for all three statements are straightforward.

The group of unit dual quaternions acts transitively on lines by conjugation, so there exist invertible $g_1, g_2 \in \mathbb{DH}$ such that $h_1 = g_1 \mathbf{i} g_1^{-1}$ and $h_2 = g_2 \mathbf{i} g_2^{-1}$. Then

$$\{q \mid q(\mathrm{i} - h_1) = 0\} = \{q \mid qg_1(\mathrm{i} - \mathbf{i})g_1^{-1} = 0\} = \{q \mid qg_1(\mathrm{i} - \mathbf{i}) = 0\} =$$

$$\mathbb{DH}_{\mathbb{C}}(\mathrm{i} + \mathbf{i})g_1^{-1} = \mathbb{DH}_{\mathbb{C}} g_1^{-1}(\mathrm{i} + h_1) = \mathbb{DH}_{\mathbb{C}}(\mathrm{i} + h_1),$$

which shows (a). The $\mathbb{D}_{\mathbb{C}}$-linear bijective map $\mathbb{DH}_{\mathbb{C}} \to \mathbb{DH}_{\mathbb{C}}$, $q \mapsto g_2^{-1} q g_1$ maps the left ideal $\mathbb{DH}_{\mathbb{C}}(\mathrm{i} + h_1)$ to the left ideal $\mathbb{DH}_{\mathbb{C}}(\mathrm{i} + \mathbf{i})$ and the right ideal $(\mathrm{i} - h_2)\mathbb{DH}_{\mathbb{C}}$ to the right ideal $(\mathrm{i} - \mathbf{i})\mathbb{DH}_{\mathbb{C}}$, which shows (b). The same map also maps the set $\{x \mid (\mathrm{i} - h_2)x(\mathrm{i} - h_1) = 0\}$ to the set $\{x \mid (\mathrm{i} - \mathbf{i})x(\mathrm{i} - \mathbf{i}) = 0\}$, which shows (c). $\square$

Lemma 2. *Assume that $l_{1,2,3} = l_{4,5,6} = 8$. Then there are at most 2 bonds $\beta := (t_1, \ldots, t_6)$ connecting h_1, h_4 for fixed values of t_1 and t_4 in $\{+\mathrm{i}, -\mathrm{i}\}$ (counted with multiplicity).*

Proof. Without loss of generality, we may assume $t_1 = t_4 = +\mathrm{i}$; the other situations can reduced to this case by replacing h_1 or h_4 or both by its negative.

By the algebraic lemma above, the intersection of the left annihilator of $(t_4 - h_4)$ and the right ideal $(t_1 - h_1)\mathbb{DH}_C$ is a 2-dimensional $\mathbb{C}$-linear subspace. Let $G \subset \mathbb{P}^7$ be its projectivization. Let $q := f_{3,6}(\beta)$ be image of a bond $\beta = (t_1, \ldots, t_6)$ connecting h_1 and h_4 with $t_1 = t_4 = +\mathrm{i}$. Then we have $q = (t_1 - h_1)(t_2 - h_2)(t_3 - h_3)$, hence q is in the right ideal $(t_1 - h_1)\mathbb{DH}_{\mathbb{C}}$. Since β connects h_1 and h_4, we have $q(t_4 - h_4) = F_{3,6}(\beta) = 0$, q is in the left annihilator of $(t_4 - h_4)$, and therefore $q \in G$.

There exist no two bonds β_1, β_2 with the same bond image q, because the parameterization of $X_{1,2,3}$ by t_1, t_2, t_3 is an isomorphism, hence q determines the first three coordinates, and the parametrization of $X_{6,5,4}$ by t_4, t_5, t_6 is also an isomorphism, hence q determines the second three coordinates. This shows that the number of bonds connecting h_1 and h_4 with $t_1 = t_4 = +\mathrm{i}$ is equal to the number of intersections of G and $C_{3,6}$; tangential intersections give rise to higher connection numbers.

On the other hand, $C_{3,6}$ is generated by quadrics, so it does not have any tritangents, so the number of such bonds is at most 2. $\square$

4. Bounding the Genus

In this section, we prove that the genus of the configuration curve of a closed 6R linkage is at most 5.

Let $L = (h_1, \ldots, h_6)$ be a closed 6R linkage with mobility 1. We use the notation of the previous section. As before, we assume that the configuration curve K has only one irreducible one-dimensional component. We write $g(K)$ for the genus of this component.

Here is an auxiliary Lemma.

Lemma 3. *Let C_1, C_2 be two curves of genus at most 1. Let $C \subset C_1 \times C_2$ be an irreducible curve such that the two projections restricted to C are either birational or 2:1 maps to C_1 resp. C_2. Then $g(C) \leq 5$, with equality only if $g(C_1) = g(C_2) = 1$ and both projections being 2:1.*

Proof. If one of the two projections is birational, say the projection to C_1, then $g(C) = g(C_1) \leq 1$. So we may assume both projections are 2:1 maps.

If C_1 and C_2 are isomorphic to $\mathbb{P}^1$, then C is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree 2, which has arithmetic genus 1. The geometric genus is 1 in the nonsingular case and 0 if C has a double point.

If $C_1 = \mathbb{P}^1$ and C_2 is elliptic, then the numerical class group is generated by the two fibers $F_1 \cong C_2$ and $F_2 \cong C_1$ of the two projections. The class of C is $2F_1 + 2F_2$, and the canonical class is $-2F_2$. Hence the arithmetic genus of C is $\frac{C(C+K)}{2} + 1 = 2(F_1 + F_2)F_1 + 1 = 3$.

If C_1 and C_2 are elliptic, then the canonical class of $C_1 \times C_2$ is zero. If F_1, F_2 are fibers of the projections, then $F_1 C = F_2 C = 2$ and $(F_1 + F_2)^2 = 2$. By the Hodge index theorem, $(C - 2F_1 - 2F_2)^2 \leq 0$, which is equivalent to $C^2 \leq 8$.

Hence the arithmetic genus of C is at most $\frac{C^2}{2} + 1 = 5$. □

Lemma 4. *If $l_{1,2,3} = 4$, then $g(K) \leq 5$.*

Proof. Let $C_1, C_2 \subset (\mathbb{P}^1)^3$ be the projections of K to (t_1, t_2, t_3) and (t_4, t_5, t_6), respectively. Let $p_1 : K \to C_1$ and $p_2 : K \to C_2$ be the projection maps. The coupler curve $C_{3,6}$ is a common image of C_1 and C_2, by the two sides of the closure equation

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3) \equiv (t_6 - h_6)(t_5 - h_5)(t_4 - h_4),$$

where we write $\equiv$ for equality in the projective sense, modulo scalar multiplication. Let $f_1 : C_1 \to C_{3,6}$ and $f_2 : C_2 \to C_{3,6}$ be these two maps. Then K is a component of the pullback of f_1, f_2. We distinguish several cases.

Case 1: $l_{6,5,4} = 4$. Then $C_{3,6}$ is the intersection of two linear subspaces, hence a line by the mobility 1 assumption. One can introduce an additional joint, rotational or translational, between links o_3 and o_6, and the linkage decomposes into two 4-bar linkages which are planar or spherical. The configuration curves of these two linkages are isomorphic to C_1 and C_2. The maps f_1, f_2 are restrictions of the 2:1 parametrizations of $X_{1,2,3}$ and $X_{6,5,4}$, hence they are either 2:1 or birational to the line $C_{3,6}$. Therefore p_1 and p_2 are also either 2:1 or birational. The configuration curve of a planar or spherical 4-bar linkage is the intersection curve of two quadrics (see [8, Chapter 11, § 8] for the planar and [9, § 21] for the spherical case). Hence its genus is at most 1. By Lemma 3, $g(K) \leq 5$.

Case 2: $l_{6,5,4} = 6$. Then $X_{6,5,4}$ is an intersection of quadrics in a 5-space and $X_{1,2,3}$ is a linear 3-space contained in the Study quadric. The intersection of both linear spaces is either a line or a plane, because the vector space $L_{6,5,4}$ does not contain any 4-dimensional subalgebras. Hence the intersection $C_{3,6}$ is either a line or a plane conic. If $C_{3,6}$ is a line, then we have a similar situation as before: the linkage decomposes into two 4-bar linkages, one planar or spherical and the second being a Bennett linkage. In any Bennett linkage, the maps from K to $\mathbb{P}^1$ parametrizing the 4 rotations are isomorphisms. Therefore K is isomorphic to the configuration curve of the planar or spherical component, hence $g(K) \leq 1$. If $C_{3,6}$ is a plane conic, then we decompose it into two rotational linear motions with coplanar axes. These two axes form together with h_3, h_4, h_5 a closed 5R linkage, which is known as the Goldberg 5R linkage (see [10]). Its configuration curve is rational, more precisely the coupling map to the plane conic is an isomorphism (see [3]). Hence K is isomorphic to C_1. Now $f_1 : C_1 \to C_{3,6}$ has 8 branching points (counted with multiplicity), namely the intersections of $C_{3,6}$ with the branching surface. By the Hurwitz genus formula, it follows that $g(K) = 3$; the genus may drop in case of singularities.

Case 3: $l_{6,5,4} = 8$. Then $C_{3,6}$ is a curve in a Segre embedding of $(\mathbb{P}^1)^3$ in $\mathbb{P}^7$ cut out by four hyperplane sections. This is only possible if $C_{3,6}$ is a twisted cubic. Then the lines h_4, h_5, h_6 could be re-covered from $C_{3,6}$ by factoring the cubic motion parametrized by $C_{3,6}$ described in [11]. On the other hand, $C_{3,6}$ is either a planar or spherical motion, hence the whole linkage is either planar or spherical, and this contradicts the mobility 1 assumption, as planar and spherical 6R linkages have mobility 3. So this case is impossible. □

Remark 1. *If $g(K) \geq 4$, then we are in Case 1, and the linkage is a composite of two planar or spherical 4-bar linkages with one common joint, which is removed from the 6-loop. The most general linkage of this type is Hooke's linkage [12], using two spherical linkages. The genus of its configuration curve is generically 5, but it may drop in the presence of singularities. If we take two planar RRRP linkages and remove the common translational joint, then we obtain the Sarrus linkage [13] with two triples of parallel consecutive axes. The bond diagrams of both linkages can be seen in Figure 2(a).*

Lemma 5. *If $l_{1,2,3} = l_{6,5,4} = 6$, then $g(K) \leq 5$.*

Proof. Let $V := L_{1,2,3} \cap L_{6,5,4}$. Then $4 \leq \dim(V) \leq 5$. The $\dim(V) = 6$ case is not possible by Lemma 6 in [3]. If $\dim(V) = 4$, then $C_{6,3}$ is embedded into a three dimensional projective space $\mathbb{P}^3$. The coupler varieties are defined by quadrics in $\mathbb{P}^5$, therefore the ideal of $C_{6,3}$ is generated by linear forms and quadrics, and so its genus is at most 1. The coupler map $f_{6,3}$ is birational, therefore $g(K) \leq 1$. So we may assume $\dim(V) = 5$.

By Theorem 3, the varieties $X_{1,2,3}$ and $X_{6,5,4}$ are complete intersections of two quadrics. We may assume in each case that one of the defining equations is the equation of the Study quadric. Then the coupler curve $C_{6,3} = X_{1,2,3} \cap X_{6,5,4}$ is defined by three quadratic equations and the linear forms defining V. It follows that $C_{6,3}$ is a complete intersection of three quadrics in $\mathbb{P}^4$, which implies $g(K) \leq 5$, with equality in the case that there are no singularities. □

Remark 2. *In [14], Dietmaier found a new linkage by a computer-supported numerical search. It turns out, by comparing the geometric parameters, that his family is exactly the family of linkages with $l_{1,2,3} = l_{4,5,6} = 6$ and $\dim(L_{1,2,3} \cap L_{6,5,4}) = 5$. See Figure 2(b) for the bond diagram of the Dietmaier linkage.*

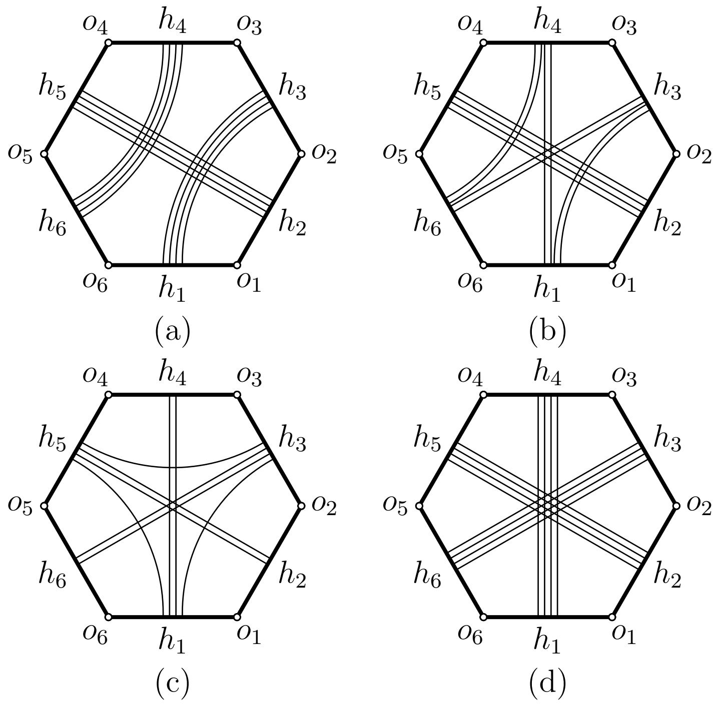

Figure 2: Bond diagrams for Hooke's double spherical linkage (a), Dietmaier's linkage (b), Wohlhart's partially symmetric linkage (c), and Bricard's orthogonal linkage (d).

Lemma 6. *If $l_{1,2,3} = 6$ and $l_{6,5,4} = 8$, then $g(K) \leq 3$.*

Proof. If $Y := X_{6,5,4} \cap L_{1,2,3}$ has dimension 1, then its Betti table coincides with the Betti table of $X_{6,5,4}$ and it follows that Y is a union of curves with genus at most 1 (the genus 1 case occurs only if Y is irreducible). Since $C_{6,3} \subseteq Y$, it follows that $g(C_{6,3}) \leq 1$, and by birationality of $f_{6,3}$ we get $g(K) \leq 1$.

Assume Y is a surface. The preimage Z of Y under the parametrization $p : (\mathbb{P}^1)^3 \to X_{6,5,4}$ is defined by two equations of tri-degree $(1,1,1)$, and because Y is a surface, the two equations must have a common divisor F which defines Z. Up to permutation of coordinates, the tri-degree of F is either $(1,0,0)$ or $(1,1,0)$. In the first case, one of the angles would be constant throughout the motion. Hence the 6R linkage is actually a 5R linkage with an extra immobile axis somewhere; then $g(K) = 0$ by the classification of 5R linkages (see [3]) (if one does not want to exclude this degenerate case). In the second case, we consider the preimage C' of $C_{6,3}$ under p. It is defined by F and the pullback of the quadric equations which defines $X_{1,2,3}$. Hence C' is a component of the complete intersection of two equations, with tri-degree $(1,1,0)$ and $(2,2,2)$. Using the first equation, we can express the first variable by the second, and so we get an isomorphic image of C' in $(\mathbb{P}^1)^2$ of bi-degree $(4,2)$, which has arithmetic genus 3. But p is is an isomorphism by Theorem 3, hence $g(C_{6,3}) \leq 3$

and $g(K) \leq 3$. $\square$

Remark 3. *An example of a linkage where* Y *is a surface is Wohlhart's partially symmetric linkage [15] (see Figure 2(c) for the bond diagram). We do not know if there exist also other linkages with* $l_{1,2,3} = 6$ *and* $l_{6,5,4} = 8$ *and* $g(K) = 3$.

Lemma 7. *If* $l_{i,i+1,i+2} = 8$ *for* $i = 1, \ldots, 6$*, then* $g(K) \leq 5$.

Proof. By Theorem 2, all bonds connect opposite joints: the bond diagram consists of b_1 connections between h_1 and h_4, b_2 connections between h_2 and h_5, and b_3 connections between h_3 and h_6. By Theorem 1, the degree of $f_{6,1}$ and the degree of $f_{3,4}$ are both equal to b_1. Note that $f_{6,1}$ and $f_{3,4}$ are the projections from K to the first and to the fourth coordinate, respectively, up to isomorphic parameterization of the line describing rotations around h_1 and h_4, respectively. Similarily, the projections to t_2, t_5, t_3, t_6 have degree b_2, b_2, b_3, b_3.

Let b_1^+ be the number of pairs of complex conjugate bonds connecting h_1 and h_4 such that $t_1 = t_4$ and b_1^- be the number of pairs such that $t_1 = -t_4$ (recall that $t_1^2 = t_4^2 = -1$). The numbers $b_2^+, b_2^-, b_3^+, b_3^-$ are defined analogously. By Lemma 2, we have $b_1^+, \ldots, b_3^- \leq 2$.

We consider the the projection $q_{1,4} : K \to (\mathbb{P}^1)^2, (t_1, \ldots, t_6) \mapsto (t_1, t_4)$. The image of this curve has bi-degree (r_1, r_1), with $r_1 \deg(q_{1,4}) = b_1$. The preimage of $(\pm\mathrm{i}, \pm\mathrm{i})$ consists entirely of bonds; moreover, if one of the coordinates of a point on $C_{1,4}$ is equal to $\pm\mathrm{i}$, then it must already be a bond. If, say, $b_1^+ = b_2^+ = 2$, and $q_{1,4}$ is birational, then $+\mathrm{i}$ is a branching point for both projections, hence it must be a double point. If $q_{1,4}$ is not birational, then it is a 2:1 map, because the preimage of any of the 4 points $(\pm\mathrm{i}, \pm\mathrm{i})$ is at most 2. In this case, the numbers b_1^+ and b_1^- are either 0 or 2, and the bi-degree of $C_{1,4}$ is $(1, 1)$ or $(2, 2)$. It follows that, in the 2:1 case, the curve $C_{1,4}$ has genus 0 or 1. We now have to sort out several cases.

Case 1: the three maps $q_{1,4}$, $q_{2,5}$ and $q_{3,6}$ are 2:1 maps. It is not possible that all three maps factor through the same 2:1 quotient, because K is contained in the product $C_{1,4} \times C_{2,5} \times C_{3,6}$. Assume, without loss of generality, that $q_{1,4}$ and $q_{2,5}$ do not factor by the same 2:1 quotient. Then $(q_{1,4}, q_{2,5}) : K \to C_{1,4} \times C_{2,5}$ is birational. By Lemma 3, the image has genus at most five, and therefore $g(K) \leq 5$.

For the remaining cases, we may assume that $q_{1,4}$ is birational.

Case 2: $b_1 = 3$. Then the arithmetic genus of $C_{1,4}$ is $(b_1 - 1)^2 = 4$. Since $b_1^+ + b_1^- = 3$, at least one of the two numbers is equal to two; assume, without loss of generality, that $b_1^+ = 2$ and $b_1^- = 1$. Then $(+\mathrm{i}, +\mathrm{i})$ and $(-\mathrm{i}, -\mathrm{i})$ are double points of $C_{1,4}$, and therefore $g(K) \leq 4 - 2 = 2$.

Case 3: $b_1 \leq 2$. Then the arithmetic genus of $C_{1,4}$ is $(b_1 - 1)^2 \leq 1$.

Case 4: $b_1 = 4$, hence $b_1^+ = b_1^- = 2$. Then the arithmetic genus of $C_{1,4}$ is $(b_1 - 1)^2 = 9$, and all four points $(\pm\mathrm{i}, \pm\mathrm{i})$ are double points. Then $g(C_{1,4}) \leq 9 - 4 = 5$, and therefore $g(K) \leq 5$. $\square$

Corollary 1. *The maximal genus 5 is reached in Case 1 when all* $C_{1,4}$*,* $C_{2,5}$*, and* $C_{3,6}$ *are elliptic and have bi-degree* $(2, 2)$*, and in Case 3; in both cases, we have* $b_1 = b_2 = b_3 = 4$.

As a consequence of Lemma 4, Lemma 5, Lemma 6, and Lemma 7 above, we finally obtain our bound for the genus.

Theorem 4. *The genus of the configuration curve of a closed 6R linkage is at most 5.*

By re-examining the proof of Lemma 7 more closely, we can prove the following theorem which will be useful later for classifying linkages with a genus 5 configuration curve.

Theorem 5. *If the bond diagram is different from the diagrams Figure 2(a), (b), and (d), then $g(K) \leq 3$.*

Proof. In view of Remark 1, Lemma 6, and the proofs of Lemmas 7 and 5, we just need to consider the case where $l_{i,i+1,i+2} = 8$ for $i = 1, \ldots, 6$. Assume indirectly that $b_1 < 4$ (using the notation as in the proof of Lemma 7). If $q_{1,4}$ is birational, then it follows $g(K) \leq 3$, hence we may assume that $q_{1,4}$ is a 2:1 map. Hence $b_1 = 2$ and $C_{1,4}$ is a curve of bi-degree $(1,1)$, which is rational. Consequently K is hyperelliptic (or $g(K) \leq 1$ and the proof is finished).

If the other two maps $q_{2,5}$, $q_{3,6}$ are also 2:1 maps, then we have a 2:1 map from K to a rational curve and another 2:1 map to a curve of genus at most 1; by Lemma 3, we obtain $g(K) \leq 3$.

So we may assume there is another map, say $q_{2,5} : K \to C_{2,5} \subset (\mathbb{P}^1)^2$, which is birational. Its image has then bi-degree (b_2, b_2), and if $b_2 \leq 3$ then we again get $g(K) \leq 3$. So we assume $b_2 = 4$. Then $C_{2,5}$ has bi-degree $(4,4)$ and 4 double points $(\pm\mathrm{i}, \pm\mathrm{i})$. The canonical map of $C_{1,4}$ is defined by the polynomials of bi-degree $(2,2)$ passing to all m-fold singular points with order $m-1$. If there is at most one double point, then it would just pass to the 4 double points $(\pm\mathrm{i}, \pm\mathrm{i})$ and maybe one additional double point, but this map maps $(\mathbb{P}^1)^2$ birational to a rational surface, and this contradicts to the fact that $C_{1,4}$ is hyperelliptic, because the canonical map of a hyperelliptic curve is 2:1. Hence there must be at least two more double points or a triple point on $C_{2,5}$, and so $g(K) \leq 3$. $\square$

Remark 4. *If the configuration curve has more than one one-dimensional component, then one can define bonds for the individual components. These bonds add up to a diagram which satisfies the same conditions we just proved for bond diagrams of irreducible configuration curves. We conclude that the genus of any component is at most 3 in a linkage with more than one component.*

5. Quad Polynomials

In this section, we recall a technique to derive algebraic equations on the parameters of a linkage from the existence of bonds connecting opposite edges, as introduced in [5].

Let $L = (h_1, \ldots, h_6)$ be a linkage, and let $\beta = (t_1, \ldots, t_6)$ be a bond connecting h_1 and h_4. We assume that $l_{1,2,3} = l_{6,5,4} = 8$, and we fix t_1 and t_4 (e.g. $t_1 = t_4 = +\mathrm{i}$). Let $G \subset \mathbb{P}^7$ be the line corresponding to the two-dimensional intersection of the left annihilator of $(t_4 - h_4)$ and the right ideal $(t_1 - h_1)\mathbb{DH}_{\mathbb{C}}$ (see Lemma 1). The intersection of G and $X_{1,2,3}$ can be computed by solving the vector-valued equation $(\mathrm{i} - h_1)(t_2 - h_2)(t_3 - h_3)(\mathrm{i} - h_4) = 0$ for t_2, t_3. Geometrically, this is the intersection of a quadric surface with the linear subspace $\{x \mid (t_1 - h_1)x(t_4 - h_4) = 0\}$. By Lemma 1, this subspace has codimension 2. So the intersection is either G or zero-dimensional of degree 2. We can exclude the first case, because the lines on $X_{1,2,3}$ appear in three well-known families (two of the three parameters being constant), and none of these families may contain G. Hence there is a quadric univariate polynomial parametrizing the intersection of G and $X_{1,2,3}$. Similarily, there is another univariate quadric polynomial parametrizing the intersection of G and $X_{6,5,4}$. The number of bonds connecting h_1 and h_4 is then bound above by the degree of the greatest common divisor of these two polynomials.

We describe this idea more concretely. Let h_1, h_2, h_3, h_4 be lines such that $l_{1,2,3} = 8$. We define the *quad polynomial* $Q_{h_1,h_2,h_3,h_4} \in \mathbb{C}[x]$ as the unique normed generator of the elimination ideal $\mathbb{C}[x] \cap I$, where $I \subset \mathbb{C}[x, y, t_2, t_3]$ is the ideal generated by the coordinates of $(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)$ and of $(t_1 - h_1)(t_2 - h_2)(t_3 - h_3) - y(1 + x\epsilon)(+\mathrm{i} - h_1)(+\mathrm{i} + h_4)$. (Here we assume that h_1 and h_4 are not parallel, which implies that $(+\mathrm{i} - h_1)(+\mathrm{i} + h_4)$ generates G as a $\mathbb{D}$-module; in the special case when h_1 and h_4 are parallel, we have to choose a different generator.)

In the following, we frequently write $a \equiv b$ for equality modulo multiplication by a nonzero complex scalar (i.e. projective equality, or both sides equal to zero).

Remark 5. *The degree of the quad polynomial is 2 unless there is a common intersection point of the coupling variety $X_{1,2,3}$, the line G, and the linear 3-space consisting of all multiples of ϵ. A closer analysis shows that the existence of such an intersection point is equivalent to either h_1 being parallel to h_2 or h_3 being parallel to h_4.*

Theorem 6. *The number of bonds connecting h_1 and h_4 with $t_1 = +\mathrm{i}$ and $t_4 = +\mathrm{i}$, counted with multiplicity, is less than or equal to the degree of the greatest common divisor of Q_{h_1,h_2,h_3,h_4} and Q_{h_4,h_5,h_6,h_1}.*

The number of bonds connecting h_1 and h_4 with $t_1 = +\mathrm{i}$ and $t_4 = -\mathrm{i}$, counted with multiplicity, is less than equal to the degree of the greatest common divisor of $Q_{h_1,h_2,h_3,-h_4}$ and Q_{-h_4,h_5,h_6,h_1}.

Proof. Let $\beta = (+\mathrm{i}, t_2, t_3, +\mathrm{i}, t_5, t_6)$ be a bond connecting h_1 and h_4. Then there exists a $z \in \mathbb{C}$ such that $f_{63}(\beta) \equiv (1 + z\epsilon)(+\mathrm{i} - h_1)(+\mathrm{i} + h_4)$ in $\mathbb{P}^7$, and so $Q_{h_1,h_2,h_3,h_4}(z) = 0$. The image $f_{3,6}(\beta)$ in $X_{4,5,6}$ is the quaternion conjugate of $f_{63}(\beta)$, which is equal to $(1 + z\epsilon)(+\mathrm{i} - h_4)(+\mathrm{i} + h_1)$. Hence $Q_{h_4,h_5,h_6,h_1}(z) = 0$. So a bond gives rise to a common root of Q_{h_1,h_2,h_3,h_4} and Q_{h_4,h_5,h_6,h_1}. Also, a bond with connection number two give rise to a common double root.

The second statement can be reduced to the first by replacing h_4 by its negative. ∎

Remark 6. *Of course there is an analogous statement for bonds with $t_1 = -\mathrm{i}$. However, we do not need these, because these bonds are complex conjugate to* *bonds with $t_1 = +\mathrm{i}$. Indeed, it is straightforward to show that $Q_{-h_1,h_2,h_3,-h_4}$ is the complex conjugate of Q_{h_1,h_2,h_3,h_4} and Q_{-h_1,h_2,h_3,h_4} is the complex conjugate of $Q_{h_1,h_2,h_3,-h_4}$. If we replace h_2 or h_3 by its negative, then the quad polynomial remains the same.*

Remark 7. *The argument of the proof of Theorem 6 can be partially reversed: a common root of the quad polynomials Q_{h_1,h_2,h_3,h_4} and Q_{h_4,h_5,h_6,h_1} implies a common point of $X_{1,2,3}$ and $X_{6,5,4}$ with norm zero. Its preimage $\alpha \in (\mathbb{P}^1)^6$ satisfies the equation $(t_1 - h_1) \ldots (t_6 - h_6) = 0$. But α is not necessarily a bond, because it also could be an isolated intersection point of $X_{1,2,3}$ and $X_{6,5,4}$, and then it is not an element in the Zariski closure of K.*

When the linkage moves, then the position of the lines change, and therefore also the quad polynomial changes. On the other hand, it is possible to describe a linkage by its Denavit–Hartenberg parameters (see [6]). For $i = 1, \ldots, 6$, ϕ_i is defined as the angle of the direction vectors of the directed lines h_i and h_{i+1}. Since this angle is determined up to sign, we require $0 \le \phi_i < \pi$. We also set $c_i := \cos(\phi)$ and $w_i = \cot(\frac{\phi_i}{2}) = \frac{cos(\phi_i)+1}{sin(\phi_i)}$.

For $i = 1, \ldots, 6$, d_i is defined as the orthogonal distance of the lines h_i and h_{i+1}. The sign of d_i is not well-defined (it would depend on an orientation of the common normal, which we do not like to choose); we will discuss the ambiguity when it arises.

If the lines h_i, h_{i+1} are not parallel, then we define the Bennett ratios as $b_i := \frac{d_i}{\sin(\phi_i)}$. (We mean no conflict with the bond number introduced in the Proof of Lemma 8; actually, we will not use these bond numbers from now on.) The sign of these numbers is well-defined: if the scalar part of $h_i h_{i+1}$ is written as $u + v\epsilon \in \mathbb{D}$, then $c_i = -u$ and $b_i = \frac{-v}{1-u^2}$.

If the lines h_i, h_{i+1} are not parallel and h_i, h_{i-1} are not parallel, then s_i is defined as the distance of the intersections of h_i and the common normals of h_i and $h_{i\pm1}$ (this parameters are called offsets). The sign of the offset is well-defined, because the two points lie on an oriented line induced by h_i.

It is well-known that the invariant parameters $c_1, \ldots, c_6, b_1, \ldots, b_6, s_1, \ldots, s_6$ form a complete system of invariants for all closed 6R linkages without adjacent parallel lines. In other words, if two such linkages share all parameters, then there is a collection of rotations in the configuration of one of them that transform it into the second. An extension to linkages with adjacent parallel lines is also well-known, but more technical. In this paper, we will assume from now on that there are no parallel adjacent lines. (As a consequence, the quad polynomials are always quadratic.)

Changing the orientation of h_i has the following effect on the parameters: $c_i, b_i, c_{i-1}, b_{i-1}, s_i$ are multiplied by -1, and all other parameters stay the same.

Remark 8. *The condition $l_{1,2,3} = 4$ is equivalent to ($b_1 = b_2 = 0$ and $s_2 = 0$), which is equivalent to the statement that the lines h_1, h_2, h_3 meet in a common point. (If we had not exluded adjacent parallel lines, then $c_1^2 = c_2^2 = 1$ would be a second possibility.)*

The condition $l_{1,2,3} = 6$ is equivalent to ($b_1^2 = b_2^2 \neq 0$ and $s_2 = 0$) (see [3], Theorem 1). Indeed, ($b_1^2 = b_2^2$ and $s_2 = 0$) is Bennett's characterization [16] of three skew lines for the existence of a fourth line forming a Bennett 4R linkage. This is the reason why the numbers $b_1, \ldots, b_6$ are called "Bennett ratios".

It is well-known that the closure equation can be expressed in terms of the Denavit-Hartenberg parameters:

$$(t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2 \cdots (t_6 - \mathbf{i})g_6 \in \mathbb{R}^*, \tag{4}$$

where

$$g_i = \left(1 - \frac{s_i}{2}\epsilon\mathbf{i}\right)(w_i - \mathbf{k})\left(1 - \frac{d_i}{2}\epsilon\mathbf{k}\right), \tag{5}$$

for $i = 1, \ldots, 6$. If one compares with (Equation 1), one recognizes constant middle factors $g_1, \ldots, g_6$ that correspond to changes of coordinate systems. Each link has an own coordinate systems where one of the axis is equal to the first coordinate axis.

It is not difficult to adapt the theory of bonds and quad polynomials to this formulation by inserting the middle factors at their proper places. For instance, the invariant quad polynomial Q_1^+ is the unique normed generator of the elimination ideal $\mathbb{C}[x] \cap I$, where $I \subset \mathbb{C}[x, y, t_2, t_3]$ is the ideal generated by the coordinates of $(t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2(t_3 - \mathbf{i})g_3(t_4 - h_4)$ and of $(t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2(t_3 - \mathbf{i}) - y(1 + x\epsilon)(\mathbf{j} + \mathrm{i}\mathbf{k})$. Using the computer algebra system Maple and resultants for eliminating t_1 and t_2, we obtain

$$Q_1^+(x) = \left(x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}\mathrm{i}\right)^2 + \frac{\mathrm{i}}{2}\left(b_1s_2 + b_3s_3 + s_2b_3c_2 + s_3b_1c_2\right) - \frac{b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2c_2^2}{4}.$$

For $i = 2, \ldots, 6$, we define the quad polynomial $Q_i^+(x)$ by a cyclic shift of indices that shifts 1 to i. Finally, we define $Q_i^-(x)$ by replacing the parameters $c_1, \ldots, c_6, b_1, \ldots, b_6$ and s_2, s_4, s_6 by their negatives, and leaving s_1, s_3, s_5 as they are. For instance,

$$Q_1^-(x) = \left(x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}\mathrm{i}\right)^2 + \frac{\mathrm{i}}{2}\left(b_1s_2 - b_3s_3 - s_2b_3c_2 + s_3b_1c_2\right) - \frac{-b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2c_2^2}{4}.$$

The full computation can be downloaded for checking from `http://people.ricam.oeaw.ac.at/z.li/softwares/quadpolynomials.html`.

The proof of Theorem 6 is still valid with middle factors inserted. Hence we get the following formulation of Theorem 6 in terms of Denavit/Hartenberg parameters.

Theorem 7. *Let $k \in \{1, \dots, 6\}$. The number of bonds connecting h_k and h_{k+3} with $t_k = +\mathrm{i}$ and $t_{k+3} = +\mathrm{i}$, counted with multiplicity, is less than or equal to the degree of the greatest common divisor of Q_k^+ and Q_{k+3}^+. Similarily, the number of bonds connecting h_k and h_{k+3} with $t_k = +\mathrm{i}$ and $t_{k+3} = -\mathrm{i}$, counted with multiplicity, is less than or equal to the degree of the greatest common divisor of Q_k^- and Q_{k+3}^-.*

Using Theorem 7, we can formulate a necessary condition for the existence of a bond connecting h_1 and h_4: either the resultant of Q_1^+ and Q_4^+ or the resultant of Q_1^- and Q_4^- has to vanish. Both resultants can be expressed as polynomials in $b_1, \dots, s_6$, but this polynomial turns out to be relatively complicated. Fortunately one obtains an easier system of equations when the maximal number of bonds is assumed.

6. Linkages with Maximal Genus

In this section we give a classification of all closed 6R linkages with a configuration curve of genus at least four that do not have links with parallel joint axes, in terms of their Denavit–Hartenberg parameters. It turns out there are four irreducible families; two of them are well-known, the other two are new.

As in the previous section, we use the angle cosines $c_1, \dots, c_6$, the Bennett ratios $b_1, \dots, b_6$ and the offsets $s_1, \dots, s_6$. In addition, we also use the f-values $f_k = c_k b_k$, $k = 1, \dots, 6$; this leads to shorter formulas.

Let L be a linkage such that no adjacent axes are parallel, and assume that the genus of its configuration curve at least four. By Theorem 5, its bond diagram is Figure 2(a), (b), or (d). Cases (a) and (b) are well-known and have been described in the Lemmas 4 and 5: these are the Hooke linkage and the Dietmaier linkage, respectively.

Remark 9. *Just for the sake of completeness, here is the description in terms of the Denavit–Hartenberg parameters (see [14]).*

Hooke linkage: $b_1 = b_2 = b_4 = b_5 = s_2 = s_5 = 0$, $d_3^2 + s_3^2 + s_4^2 - 2c_3 s_2 s_4 = d_6^2 + s_2^2 + s_5^2 - 2c_6 s_1 s_5$.

Dietmaier linkage: $b_1 = b_2, b_4 = b_5, b_3 = b_6, c_3 = c_6, f_1 + f_2 = f_4 + f_5, s_1 = s_3, s_4 = s_6, s_2 = s_5 = 0$ *up to orientation of the axes.*

From now on, we assume that $l_{k,k+1,k+2} = 8$ for $k = 1, \dots, 6$; consequently, the bond diagram is Figure 2(d). The number of bonds is maximal, for $k = 1, 2, 3$, and for any choice of t_k, t_{k+3} in $\{+\mathrm{i}, -\mathrm{i}\}$, there exist 2 bonds connecting h_k and h_{k+3}. By Theorem 7, we get the following equalities of polynomials in $\mathbb{C}[x]$:

$$Q_1^+ = Q_4^+, Q_2^+ = Q_5^+, Q_3^+ = Q_6^+, Q_1^- = Q_4^-, Q_2^- = Q_5^-, Q_3^- = Q_6^-. \tag{6}$$

Each equality of polynomials gives rise to four scalar equations, namely the real and imaginary part of the linear and the constant coefficient.

Lemma 8. *The zero set of the 24 equations above is the union of two irreducible components. For both, we have* $s_1 = \cdots = s_6 = 0$ *and the three equations*

$$b_1c_2b_3 = b_4c_5b_6, b_2c_3b_4 = b_5c_6b_1, b_3c_4b_5 = b_6c_1b_2. \tag{7}$$

The two components are

1. $f_1 = f_4, f_2 = f_5, f_3 = f_6, b_1b_3b_5 = b_2b_4b_6,$
   $b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2$
2. $f_1 = f_3 = f_5, f_2 = f_4 = f_6, b_1b_3b_5f_2 = b_2b_4b_6f_1,$
   $b_1^2 + b_3^2 + b_5^2 + f_2^2 = b_2^2 + b_4^2 + b_6^2 + f_1^2.$

If no Bennett ratio is zero, then the three equations (7) are redundant.

Proof. By comparing the imaginary parts of the linear coefficients, it follows immediately that $s_1 = \cdots = s_6 = 0$. For the simplified system, we obtained the decomposition above by Gröbner basis computation using the computer algebra system Maple. □

Theorem 8. *There are two irreducible families of 6R linkages with coupling dimensions 8 such that the configuration curve has genus 5 generically. They are characterized by cases 1 and 2 in Lemma 8.*

Proof. The validity of the equations (6) implies the existence of 24 points in the intersection of $X_{1,2,3}$ and $X_{6,5,4}$, by Remark 7. Intersection theory predicts an intersection of only 16 points (see [7, Section 11.5.1]), therefore the intersection is infinite and the linkage moves.

Since the genus is a lower semicontinuous function in a family of curves, and 5 is the largest possible value, it suffices to exhibit a single example with a configuration curve of genus 5 for each of the two families in order to prove that the genus is 5 in the generic case. Here is an example that works for both, because it is in the intersection of the two families:

$$b_1 = 0, b_2 = 40, b_3 = 32, b_4 = 0, b_5 = 25, b_6 = 7, c_1 = \cdots = c_6 = 0.$$

□

Remark 10. *A special case of the second family is Bricard's orthogonal linkage (see [17]). It can be characterized by the condition* $s_1 = \cdots = s_6 = c_1 = \cdots = c_6 = 0$ *and* $b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2$. *The example in the proof of Theorem 7 is actually an instance of Bricard's orthogonal linkage. Therefore we can conclude that the genus of the configuration curve of Bricard's orthogonal linkage is 5 generically.*

Remark 11. *The linkages with a configuration curve of genus 4 are contained in the 4 families described in this section as special cases. A concrete example is the Bricard orthogonal linkage with* $(b_1, \ldots, b_6) = (4, 3, 5, 7, 9, 8)$.

Acknowledgements

This research was supported by the Austrian Science Fund (FWF): DK W 1214-N15 and P 26607-N25.

[1] E. Delassus, Les chaines articulees fermees et deformables a quatre membres, Bulletin des Sciences Mathematiques at Astronomiques 46 (1922) 283–304.

[2] A. Karger, Classification of 5R closed kinematic chains with self mobility, Mech. Machine Theory (1998) 213–222.

[3] G. Hegedüs, J. Schicho, H.-P. Schröcker, The theory of bonds: A new method for the analysis of linkages, Mech. Machine Theory 70 (2013) 407–424. `doi:10.1016/j.mechmachtheory.2013.08.004`.

[4] G. Nawratil, Introducing the theory of bonds for Stewart-Gough platforms with self-motions, ASME J. Mech. Robotics 6.

[5] Z. Li, J. Schicho, A new technique for analyzing 6R linkages: quad polynomials, Tech. Rep. 1402.5761, ArXiV (2014).

[6] M. Husty, H.-P. Schröcker, Algebraic geometry and kinematics, in: I. Z. Emiris, F. Sottile, T. Theobald (Eds.), Nonlinear Computational Geometry, Vol. 151 of The IMA Volumes in Mathematics and its Applications, Springer, 2010, Ch. Algebraic Geometry and Kinematics, pp. 85–107.

[7] J. Selig, Geometric Fundamentals of Robotics, 2nd Edition, Monographs in Computer Science, Springer, 2005.

[8] O. Bottema, B. Roth, Theoretical Kinematics, Dover Publications, 1990.

[9] H. R. Müller, Sphärische Kinematik, VEB Deutscher Verlag der Wissenschaften, Berlin, 1962.

[10] M. Goldberg, New five-bar and six-bar linkages in three dimensions, Trans. ASME 65 (1943) 649–656.

[11] G. Hegedüs, J. Schicho, H.-P. Schröcker, Factorization of rational curves in the Study quadric and revolute linkages, Mech. Machine Theory 69 (1) (2013) 142–152. `doi:10.1016/j.mechmachtheory.2013.05.010`.

[12] J. E. Baker, Displacement-closure equations of the unspecialised double-Hooke's-joint linkage, Mech. Machine Theory 37 (2002) 1127–1142.

[13] M. Sarrus, Note sur la transformation des mouvements rectilignes alternatifs, en mouvements circulaires; et rciproquement, C. R. Akad. des Sciences (1853) 1038.

[14] P. Dietmaier, Einfach übergeschlossene Mechanismen mit Drehgelenken, Habilitation thesis, Graz University of Technology (1995).

[15] K. Wohlhart, A new 6R space mechanism, in: Proc. IFToMM 7, 1987, pp. 193–198.

[16] G. T. Bennett, The skew isogramm-mechanism, Proc. London Math. Soc. 13 (2nd Series) (1913–1914) 151–173.

[17] J. E. Baker, An analysis of Bricard linkages, Mech. Machine Theory 15 (4) (1980) 267–286.