Dual-primal isogeometric tearing and interconnecting solvers for multipatch continuous and discontinuous Galerkin IgA equations

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Abstract

We present results regarding fast and robust solvers for equations arising from continuous and discontinuous Galerkin discretization of heterogeneous diffusion problems in the context of Isogeometric Analysis. The solvers considered belong to the class of non-overlapping domain decomposition methods which use tearing and interconnection strategies.

1 Introduction

In this paper, we consider the adaption of Dual Primal Finite Element Tearing and Interconnecting (FETI-DP) methods to Isogeometric Analysis (IgA). This class of IgA solvers are now called Dual Primal Isogeometric Tearing and Interconnecting (IETI-DP) methods. We apply these methods to large-scale linear systems of algebraic equations arising from continuous Galerkin (cG) and discontinuous Galerkin (dG) IgA of heterogeneous diffusion problems on multipatch domains. The dG formulation is used to couple the local problems across patch interfaces with possibly non-matching grids. The purpose of this paper is to present cG and dG IETI-DP methods, summarize some theoretical results, and discuss our numerical results.

2 Multipatch IgA of heterogeneous diffusion problems

For sake of simplicity, let us consider the following dG IgA discretization of the homogeneous Dirichlet problem for the diffusion equation $-\text{div}(\alpha\nabla u) = f$ in a multipatch domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a non-homogeneous diffusion coefficient α and a given source term f. The computational domain Ω is decomposed into N non-overlapping subdomains Ω_k, called volumetric patches in IgA. Every patch Ω_k is the image of the parameter domain $(0,1)^d$ by a regular NURBS-map $G^{(k)} : [0,1]^d \to \overline{\Omega}_k$, see [1] for an introduction to the IgA technology. Now the dG IgA scheme, appoximating our diffusion problem, can be formulated as follows: Find $u_h \in V_h = \prod_k \text{span}\{N_{i_k,p_k} \circ {G^{(k)}}^{-1} : \text{supp}(N_{i_k,p_k}) \cap \partial\Omega = \emptyset\}$ such that

$$\sum_{k=1}^{N} a_k(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} f v_k dx \quad \forall v \in V_h, \tag{1}$$

where $a_k(u,v) = \int_{\Omega_k} \alpha_k \nabla u_k \cdot \nabla v_k dx + \sum_{l\in\mathcal{E}_k} \int_{\Gamma_{kl}} \frac{\alpha_k}{2} \left(\frac{\partial u_k}{\partial n}(v_l - v_k) + \frac{\partial v_k}{\partial n}(u_l - u_k)\right) + \frac{\mu\alpha_k}{h}(u_l - u_k)(v_l - v_k) ds$. Here we assume that $\alpha = \alpha_k = const > 0$ in Ω_k. The cG IgA scheme can be seen as special case of (1), where we look for a continuous IgA approximation $u_h \in V_{0h} := V_h \cap H_0^1(\Omega)$ solving (1) for all test functions $v_h \in V_{0h}$. The NURBS functions $\{N_{i_k,p_k}\}$ are used as basis functions for constructing the geometrical map $G^{(k)}$ as well as for construction the IgA approximation u_h to the solution u of our diffusion problem on the patch Ω_k.

3 Tearing and Interconnecting

The idea of Tearing and Interconnecting consists in the introduction of additional dofs on the interface in order to decouple the local subproblems and to ensure the continuity of the solution by means Lagrange multipliers.

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The operator that enforces the continuity is usually called *Jump Operator* and is denoted with B. For the cG IgA scheme, we follow the standard procedure as described, e.g., in [2]. However, it is not obvious how to apply this technique to the dG IgA scheme (1). Following the dG FETI-DP technique proposed in [3], we introduce an additional layer of dofs, which represents the opposite subdomain interface, to the interface part of each patch. This technique is in detail explained in [4]. Consequently, we can equivalently rewrite the corresponding linear system as

$$\begin{bmatrix} K_e^{cG} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_e \\ \lambda \end{bmatrix} = \begin{bmatrix} f_e^{cG} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} K_e^{dG} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_e \\ \lambda \end{bmatrix} = \begin{bmatrix} f_e^{dG} \\ 0 \end{bmatrix}, \tag{2}$$

where $K_e^X = \text{diag}_k(K_k^X)$ and $f_e^X = [f_k^X]$ with $X \in \{cG, dG\}$ are the subdomain stiffness matrices and right-hand side vectors. We introduce primal variables in order to guarantee the invertibility of K_e^X, however, at the price of loosing its block-diagonal structure. We denote entries of the system with incorporated primal variables with a tilde, i.e., $\widetilde{K}_e^X, \widetilde{B}, \widetilde{f}_e^X, \ldots$. From this system, we obtain the Schur complement equation: Find $\lambda \in U$ such that

$$F^X \lambda = d^X, \tag{3}$$

where $F^X := \widetilde{B} \left(\widetilde{K}_e^X\right)^{-1} \widetilde{B}^T$ and $d^X := \widetilde{B} \left(\widetilde{K}_e^X\right)^{-1} \widetilde{f}^X$. Clearly, this system is never explicitly generated. Instead, we apply a matrix free CG algorithm, where we just implement the application of the matrix. For an efficient implementation concerning the application of F and the incorporation of the primal variables, we refer, e.g., to [5] and [2]. We note that these techniques can also be applied to the case $X = dG$, where adaptions to the corresponding index sets have to be made.

4 Robust preconditioners

In order to obtain an efficient and robust method, we use the preconditioned conjugate gradient (PCG) algorithm for solving the Schur complement equation (3). We choose the so-called scaled Dirichlet preconditioner $M_{sD}^X := B_D S_e^X B_D$, where S_e^X is a block diagonal matrix of subdomain local Schur complements, and B_D denotes a scaled version of B. In the case of $X = cG$, it is shown in [2] that the spectral condition number of the preconditioned system is bounded by $O\left((1 + \log(H/h)^2)\right)$, where $H/h := \max_k H_k/h_k$. Thus, the bound is quasi optimal with respect to the ratio of the patch diameter H_k and the mesh size h_k. This condition number implies that the number of PCG iteration grows at most like $O\left((1 + \log(H/h))\right)$.

5 Discussion of our numerical results

The $O\left((1 + \log(H/h))\right)$ growth of the number of PCG iteration that was proved in [2] for cG IgA equations has been confirmed by all our numerical experiments, see also [2]. Moreover, our numerical results indicate robustness with respect to jumping diffusion coefficients across the interface and a $p-$dependence of order at most $O(p)$ in 2D and 3D, where p denotes the underlying polynomial degree. The analysis of the case $X = dG$ will be studied in a forthcoming paper. Fortunately, the numerical results in 2D and 3D clearly indicate the same quasi-optimal bound and the same behaviour with respect to p and jumping diffusion coefficients as in the case of cG IgA equation, see [4]. Moreover, dG IETI-DP also works for segmentations (domain decomposition) with gaps and overlaps descussed in [6], as one can see in Fig. 1 and Table 2.

Acknowledgement

This research work was supported by the Austrian Science Fund (FWF) through the NFN S117-03 project.

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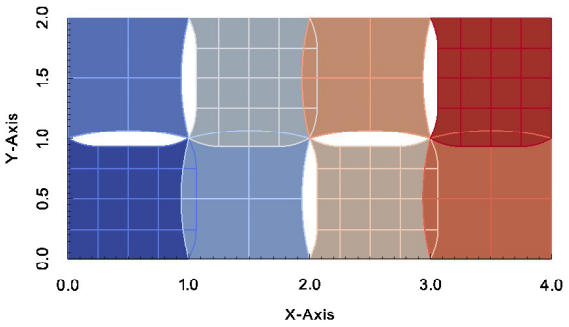

Figure 1: Segmentation with gaps and overlaps

Alg. C		coeff. scal.		stiff. scal.	
Dofs	H/h	It.	κ	It.	κ
1908	16	9	1.50	9	1.50
6348	32	10	1.77	10	1.76
22908	64	10	2.08	10	2.07
86748	128	11	2.43	11	2.42
337308	256	12	2.82	13	2.84

Figure 2: Condition number (κ) and CG iterations (It.) for coefficient and stiffness scaling; polynomial degree 2 with homogeneous diffusion coefficient.

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