

# D-finite Numbers\*

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## Abstract

D-finite functions and P-recursive sequences are defined in terms of linear differential and recurrence equations with polynomial coefficients. In this paper, we introduce a class of numbers closely related to D-finite functions and P-recursive sequences. It consists of the limits of convergent P-recursive sequences. Typically, this class contains many well-known mathematical constants in addition to the algebraic numbers. Our definition of the class of D-finite numbers depends on two subrings of the field of complex numbers. We investigate how different choices of these two subrings affect the class. Moreover, we show that D-finite numbers over the Gaussian rational field are essentially the same as the values of D-finite functions at non-singular algebraic number arguments. This result makes it easier to recognize certain numbers as D-finite.

## 1 Introduction

D-finite functions have been recognized long ago [20, 14, 25, 18, 15, 21] as an especially attractive class of functions. They are interesting on the one hand because each of them can be easily described by a finite amount of data, and efficient algorithms are available to do exact as well as approximate computations with them. On the other hand, the class is interesting because it covers a lot of special functions which naturally appear in various different contexts, both within mathematics as well as in applications.

The defining property of a *D-finite* function is that it satisfies a linear differential equation with polynomial coefficients. This differential equation, together with an appropriate number of initial terms, uniquely determines the function at hand. Similarly, a sequence is called *P-recursive* (or rarely, *D-finite*) if it satisfies a linear recurrence equation with polynomial coefficients. Also in this case, the equation together with an appropriate number of initial terms uniquely determines the object.

In a sense, the theory of D-finite functions generalizes the theory of algebraic functions. Many concepts that have first been introduced for the latter have later been formulated also for the former. In particular, every algebraic function is D-finite (Abel's theorem), and many properties the class of algebraic function enjoys carry over to the class of D-finite functions.

The theory of algebraic functions in turn may be considered as a generalization of the classical and well-understood class of algebraic numbers. The class of algebraic numbers suffers from being relatively small. There are many important numbers, most prominently the numbers  $e$  and  $\pi$ , which are not algebraic.

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Many larger classes of numbers have been proposed, let us just mention two examples. The first is the class of periods (in the sense of Kontsevich and Zagier [13]). These numbers are defined as the values of multivariate definite integrals of algebraic functions over a semi-algebraic set. In addition to all the algebraic numbers, this class contains important numbers such as  $\pi$ , all zeta constants (the Riemann zeta function evaluated at an integer) and multiple zeta values, but it is so far not known whether for example  $e$ ,  $1/\pi$  or Euler's constant  $\gamma$  are periods (conjecturally they are not). Another class of numbers is the class of all numbers that appear as values of so-called G-functions (in the sense of Siegel [19]) at algebraic number arguments [4, 5]. The class of G-functions is a subclass of the class of D-finite functions, and it inherits some useful properties of that class. Among the values that G-functions can assume are  $\pi$ ,  $1/\pi$ , values of elliptic integrals and multiple zeta values, but it is so far not known whether for example  $e$ , Euler's constant  $\gamma$  or a Liouville number are such a value (conjecturally not).

In this paper we propose a new class of numbers.

**Definition 1.** *Let  $R$  be a subring of  $\mathbb{C}$  and let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$ .*

1. *A number  $\xi \in \mathbb{C}$  is called D-finite (with respect to  $R$  and  $\mathbb{F}$ ) if there exists a convergent sequence  $(a_n)_{n=0}^{\infty}$  in  $R^{\mathbb{N}}$  with  $\lim_{n \rightarrow \infty} a_n = \xi$  and some polynomials  $p_0, \dots, p_r \in \mathbb{F}[n]$ ,  $p_r \neq 0$ , not all zero, such that*

$$p_0(n)a_n + p_1(n)a_{n+1} + \dots + p_r(n)a_{n+r} = 0$$

*for all  $n \in \mathbb{N}$ .*

2. *The set of all D-finite numbers with respect to  $R$  and  $\mathbb{F}$  is denoted by  $\mathcal{D}_{R,\mathbb{F}}$ . If  $R = \mathbb{F}$ , we also write  $\mathcal{D}_{\mathbb{F}} := \mathcal{D}_{\mathbb{F},\mathbb{F}}$  for short.*

It is clear that  $\mathcal{D}_{R,\mathbb{F}}$  contains all the elements of  $R$ , but it typically contains many further elements. For example, let  $i$  be the imaginary unit, then  $\mathcal{D}_{\mathbb{Q}(i)}$  contains many (if not all) the periods and, as we will see below, many (if not all) the values of G-functions. In addition, it is not hard to see that  $e$  and  $1/\pi$  are D-finite numbers. According to Fischler and Rivoal's work [5], also Euler's constant  $\gamma$  and any value of the Gamma function at a rational number are D-finite. (We thank Alin Bostan for pointing us to this reference.) We will show below that the limits of convergent P-recursive sequences over  $\mathbb{Q}(i)$  are essentially the same as the values D-finite functions can assume at non-singular algebraic number arguments. Together with the work on arbitrary-precision evaluation of D-finite functions [3, 22, 23, 24, 16, 17], it follows that D-finite numbers are computable in the sense that for every D-finite number  $\xi$  there exists an algorithm which for any given  $n \in \mathbb{N}$  computes a numeric approximation of  $\xi$  with a guaranteed precision of  $10^{-n}$ . Consequently, all non-computable numbers have no chance to be D-finite. Besides these artificial examples, we do not know of any explicit real numbers which are not in  $\mathcal{D}_{\mathbb{Q}}$ , and we believe that it may be very difficult to find some.

The definition of D-finite numbers given above involves two subrings of  $\mathbb{C}$  as parameters: the ring to which the sequence terms of the convergent sequences are supposed to belong, and the field to which the coefficients of the polynomials in the recurrence equations should belong. Obviously, these choices matter, because we have, for example,  $\mathcal{D}_{\mathbb{R}} = \mathbb{R} \neq \mathbb{C} = \mathcal{D}_{\mathbb{C}}$ . Also, since  $\mathcal{D}_{\mathbb{Q}}$  is a countable set, we have  $\mathcal{D}_{\mathbb{Q}} \neq \mathcal{D}_{\mathbb{R}}$ . On the other hand, different choices of  $R$  and  $\mathbb{F}$  may lead to the same classes. For example, we would not get more numbers by allowing  $\mathbb{F}$  to be a subring of  $\mathbb{C}$  rather than a field, because we can always clear denominators in a defining recurrence. One of the goals of this article is to investigate how  $R$  and  $\mathbb{F}$  can be modified without changing the resulting class of D-finite numbers.

As a long-term goal, we hope to establish the notion of D-finite numbers as a class that naturally relates to the class of D-finite functions in the same way as the classical class of algebraic numbers relates to the class of algebraic functions.

## 2 D-finite Functions and P-recursive Sequences

Throughout the paper,  $R$  is a subring of  $\mathbb{C}$  and  $\mathbb{F}$  is a subfield of  $\mathbb{C}$ , as in Definition 1 above. We consider linear operators that act on sequences or power series and analytic functions. We write  $S_n$  for the shift operator w.r.t.  $n$  which maps a sequence  $(a_n)_{n=0}^{\infty}$  to  $(a_{n+1})_{n=0}^{\infty}$ . The set of all linear operators of the form  $L := p_0 + p_1 S_n + \cdots + p_r S_n^r$ , with  $p_0, \dots, p_r \in \mathbb{F}[n]$ , forms an Ore algebra; we denote it by  $\mathbb{F}[n]\langle S_n \rangle$ . Analogously, we write  $D_z$  for the derivation operator w.r.t.  $z$  which maps a power series or function  $f(z)$  to its derivative  $f'(z) = \frac{d}{dz} f(z)$ . Also the set of linear operators of the form  $L := p_0 + p_1 D_z + \cdots + p_r D_z^r$ , with  $p_0, \dots, p_r \in \mathbb{F}[z]$ , forms an Ore algebra; we denote it by  $\mathbb{F}[z]\langle D_z \rangle$ . For an introduction to Ore algebras and their actions, see [1]. When  $p_r \neq 0$ , we call  $r$  the *order* of the operator and  $\text{lc}(L) := p_r$  its *leading coefficient*.

### Definition 2.

1. A sequence  $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$  is called P-recursive or D-finite (of order  $r$ ) over  $\mathbb{F}$  if there exists a nonzero operator  $L = \sum_{j=0}^r p_j(n) S_n^j \in \mathbb{F}[n]\langle S_n \rangle$  such that

$$L \cdot a_n = p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0$$

for all  $n \in \mathbb{N}$ .

2. A formal power series  $f(z) \in R[[z]]$  is called D-finite (of order  $r$ ) over  $\mathbb{F}$  if there exists a nonzero operator  $L = \sum_{j=0}^r p_j(z) D_z^j \in \mathbb{F}[z]\langle D_z \rangle$  such that

$$L \cdot f(z) = p_r(z) D_z^r f(z) + \cdots + p_1(z) D_z f(z) + p_0(z) f(z) = 0.$$

3. A formal power series  $f(z) \in \mathbb{F}[[z]]$  is called algebraic over  $\mathbb{F}$  if there exists a nonzero bivariate polynomial  $P(z, y) \in \mathbb{F}[z, y]$  such that  $P(z, f(z)) = 0$ .

A formal power series is D-finite if and only if its coefficient sequence is P-recursive. Many elementary functions like rational functions, exponentials, logarithms, sine, algebraic functions, etc., as well as many special functions, like hypergeometric series, error functions, Bessel functions, etc., are D-finite. Hence their respective coefficient sequences are P-recursive.

The class of D-finite functions (resp. P-recursive sequences) is closed under certain operations: addition, multiplication, derivative (resp. forward shift) and integration (resp. summation). In particular, the set of D-finite functions (resp. P-recursive sequences) forms a left- $\mathbb{F}[z]\langle D_z \rangle$ -module (resp. a left- $\mathbb{F}[n]\langle S_n \rangle$ -module). Also, if  $f$  is a D-finite function and  $g$  is an algebraic function, then the composition  $f \circ g$  is D-finite. These and further closure properties are easily proved by linear algebra arguments, proofs can be found for instance in [20, 18, 11]. We will make free use of these facts.

We will be considering singularities of D-finite functions. Recall from the classical theory of linear differential equations [10] that a linear differential equation  $p_0(z)f(z) + \cdots + p_r(z)f^{(r)}(z) = 0$  with polynomial coefficients  $p_0, \dots, p_r \in \mathbb{F}[z]$  and  $p_r \neq 0$  has a basis of analytic solutions in a neighborhood of every point  $\xi \in \mathbb{C}$ , except possibly at roots of  $p_r$ . The roots of  $p_r$  are therefore called the *singularities* of the equation (or the corresponding linear operator). If  $\xi \in \mathbb{C}$  is a

singularity of the equation but the equation nevertheless admits a basis of analytic solutions at this point, we call it an *apparent singularity*. It is well-known [10, 2] that for any given linear differential equations with some apparent and some non-apparent singularities, we can always construct another linear differential equation (typically of higher order) whose solution space contains the solution space of the first equation and whose only singularities are the non-apparent singularities of the first equation. This process is known as desingularization. For later use, we will give a proof of the composition closure property for D-finite functions which pays attention to the singularities.

**Theorem 3.** *Let  $P(z, y) \in \mathbb{F}[z, y]$  be square-free of degree  $d$ , and let  $L \in \mathbb{F}[z]\langle D_z \rangle$  nonzero. Let  $\zeta \in \mathbb{C}$  be such that  $P$  defines  $d$  distinct analytic algebraic functions  $g(z)$  with  $P(z, g(z)) = 0$  in a neighborhood of  $\zeta$ , and assume that for none of these functions, the value  $g(\zeta) \in \mathbb{C}$  is a singularity of  $L$ . Fix a solution  $g$  of  $P$  and an analytic solution  $f$  of  $L$  defined in a neighborhood of  $g(\zeta)$ . Then there exists a nonzero operator  $M \in \mathbb{F}[z]\langle D_z \rangle$  with  $M \cdot (f \circ g) = 0$  which does not have  $\zeta$  among its singularities.*

*Proof.* (borrowed from [12]) Consider the operator  $\tilde{L} := L(g, (g')^{-1}D_z) \in \overline{\mathbb{F}(z)}\langle D_z \rangle$ . It is easy to check that  $L \cdot f = 0$  if and only if  $\tilde{L} \cdot (f \circ g) = 0$  for every root  $g$  of  $P$  near  $\zeta$ . Therefore, if  $f_1, \dots, f_r$  is a basis of the solution space of  $L$  near  $g(\zeta)$ , then  $f_1 \circ g, \dots, f_r \circ g$  is a basis of the solution space of  $\tilde{L}$  near  $\zeta$ .

Let  $g_1, \dots, g_d$  be all the solutions of  $P$  near  $\zeta$ , and let  $M$  be the least common left multiple of all the operators  $L(g_j, (g'_j)^{-1}D_z)$ . Then the solution space of  $M$  near  $\zeta$  is generated by all the functions  $f_i \circ g_j$ . Since the coefficients of  $M$  are symmetric w.r.t. the conjugates  $g_1, \dots, g_d$ , they belong to the ground field  $\mathbb{F}(z)$ , and after clearing denominators (from the left) if necessary, we may assume that  $M$  is an operator in  $\mathbb{F}[z]\langle D_z \rangle$  whose solution space is generated by functions that are analytic at  $\zeta$ . Therefore, by the remarks made about desingularization, it is possible to replace  $M$  by an operator (possibly of higher order) which does not have  $\zeta$  among its singularities.  $\square$

By a similar argument, we see that algebraic extensions of the coefficient field of the recurrence operators are useless. Moreover, it is also not useful to make  $\mathbb{F}$  bigger than the quotient field of  $R$ .

**Lemma 4.**

1. If  $\mathbb{E}$  is an algebraic extension field of  $\mathbb{F}$  and  $(a_n)_{n=0}^\infty$  is  $P$ -recursive over  $\mathbb{E}$ , then it is also  $P$ -recursive over  $\mathbb{F}$ .
2. If  $R \subseteq \mathbb{F}$  and  $(a_n)_{n=0}^\infty \in R^\mathbb{N}$  is  $P$ -recursive over  $\mathbb{F}$ , then it is also  $P$ -recursive over  $\text{Quot}(R)$ , the quotient field of  $R$ .
3. If  $\mathbb{F}$  is closed under complex conjugation and  $(a_n)_{n=0}^\infty$  is  $P$ -recursive over  $\mathbb{F}$ , then so are  $(\bar{a}_n)_{n=0}^\infty$ ,  $(\text{Re}(a_n))_{n=0}^\infty$ , and  $(\text{Im}(a_n))_{n=0}^\infty$ .

*Proof.* 1. Let  $L \in \mathbb{E}[n]\langle S_n \rangle$  be an annihilating operator of  $(a_n)_{n=0}^\infty$ . Then, since  $L$  has only finitely many coefficients,  $L \in \mathbb{F}(\theta)[n]\langle S_n \rangle$  for some  $\theta \in \mathbb{E}$ . Let  $M$  be the least common left multiple of all the conjugates of  $L$ . Then  $M$  is an annihilating operator of  $(a_n)_{n=0}^\infty$  which belongs to  $\mathbb{F}[n]\langle S_n \rangle$ . The claim follows.

2. Let us write  $\mathbb{K} = \text{Quot}(R)$ . Let  $L \in \mathbb{F}[n]\langle S_n \rangle$  be a nonzero annihilating operator of  $(a_n)_{n=0}^\infty$ . Since  $\mathbb{F}$  is an extension field of  $\mathbb{K}$ , it is a vector space over  $\mathbb{K}$ . Write

$$L = \sum_{m=0}^r \sum_{j=0}^{d_m} p_{mj} n^j S_n^m,$$

where  $r, d_m \in \mathbb{N}$  and  $p_{mj} \in \mathbb{F}$  not all zero. Then the set of the coefficients  $p_{ij}$  belongs to a finite dimensional subspace of  $\mathbb{F}$ . Let  $\{\alpha_1, \dots, \alpha_s\}$  be a basis of this subspace over  $\mathbb{K}$ . Then for each pair  $(m, j)$ , there exists  $c_{mj\ell} \in \mathbb{K}$  such that  $p_{mj} = \sum_{\ell=1}^s c_{mj\ell} \alpha_\ell$ , which gives

$$0 = L \cdot a_n = \sum_{\ell=1}^s \alpha_\ell \underbrace{\left( \sum_{m=0}^r \sum_{j=0}^{d_m} c_{mj\ell} n^j a_{n+m} \right)}_{=: b_n \in \mathbb{K}}.$$

For all  $n \in \mathbb{N}$ , it follows from the linear independence of  $\{\alpha_1, \dots, \alpha_s\}$  that  $b_n = 0$ . Therefore

$$\sum_{m=0}^r \underbrace{\left( \sum_{j=0}^{d_m} c_{mj\ell} n^j \right)}_{\in \mathbb{K}[n]} S_n^m \cdot a_n = 0 \quad \text{for all } n \in \mathbb{N} \text{ and } \ell = 1, \dots, s.$$

Thus  $(a_n)_{n=0}^\infty$  has a nonzero annihilating operator with coefficients in  $\mathbb{K}[n]$ .

3. Since  $(a_n)_{n=0}^\infty$  is P-recursive over  $\mathbb{F}$ , there exists a nonzero operator  $L$  in  $\mathbb{F}[n]\langle S_n \rangle$  such that  $L \cdot a_n = 0$ . Hence  $\bar{L} \cdot \bar{a}_n = 0$  where  $\bar{L}$  is the operator obtained from  $L$  by taking the complex conjugate of each coefficient. Since  $\mathbb{F}$  is closed under complex conjugation by assumption,  $\bar{L}$  belongs to  $\mathbb{F}[n]\langle S_n \rangle$ , and hence  $(\bar{a}_n)_{n=0}^\infty$  is P-recursive over  $\mathbb{F}$ .

Because of  $\text{Re}(a_n) = \frac{1}{2}(a_n + \bar{a}_n)$  and  $\text{Im}(a_n) = \frac{1}{2i}(a_n - \bar{a}_n)$  with  $i$  the imaginary unit, the other two assertions follow by closure properties. □

Of course, all the statements hold analogously for D-finite power series instead of P-recursive sequences.

If we consider a D-finite function as an analytic complex function defined in a neighborhood of zero, then this function can be extended by analytic continuation to any point in the complex plane except for finitely many ones, namely the singularities of the given function. In this sense, D-finite functions can be evaluated at any non-singular point by means of analytic continuation. Numerical evaluation algorithms for D-finite functions have been developed in [3, 22, 23, 24, 16, 17], where the last two references also provide a MAPLE implementation, namely the NumGfun package, for computing such evaluations. These algorithms perform arbitrary-precision evaluations with full error control.

### 3 Algebraic Numbers

Before turning to general D-finite numbers, let us consider the subclass of algebraic functions. We will show that in this case, the possible limits are precisely the algebraic numbers. For the purpose of this article, let us say that a sequence  $(a_n)_{n=0}^\infty \in \mathbb{F}^\mathbb{N}$  is *algebraic* over  $\mathbb{F}$  if the

corresponding power series  $\sum_{n=0}^{\infty} a_n z^n \in \mathbb{F}^{\mathbb{N}}$  is algebraic in the sense of Definition 2. Since algebraic functions are D-finite, it is clear that algebraic sequences are P-recursive. We will write  $\mathcal{A}_{\mathbb{F}}$  for the set of all numbers  $\xi \in \mathbb{C}$  which are limits of convergent algebraic sequences over  $\mathbb{F}$ .

Recall [8] that two sequences  $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$  are called *asymptotically equivalent*, written  $a_n \sim b_n$  ( $n \rightarrow \infty$ ), if the quotient  $a_n/b_n$  converges to 1 as  $n \rightarrow \infty$ . Similarly, two complex functions  $f(z)$  and  $g(z)$  are called *asymptotically equivalent* at a point  $\zeta \in \mathbb{C}$ , written  $f(z) \sim g(z)$  ( $z \rightarrow \zeta$ ), if the quotient  $f(z)/g(z)$  converges to 1 as  $z$  approaches  $\zeta$ . These notions are connected by the following classical theorem.

**Theorem 5.**

1. (Transfer theorem [7, 8]) For every  $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$  we have

$$[z^n] \frac{1}{(1-z)^\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad (n \rightarrow \infty),$$

where  $\Gamma(z)$  stands for the Gamma function and the notation  $[z^n]f(z)$  refers to the coefficient of  $z^n$  in the power series  $f(z) \in \mathbb{F}[[z]]$ .

2. (Basic Abelian theorem [6]) Let  $(a_n)_{n=0}^{\infty} \in \mathbb{F}^{\mathbb{N}}$  be a sequence that satisfies the asymptotic estimate

$$a_n \sim n^\alpha \quad (n \rightarrow \infty),$$

where  $\alpha \geq 0$ . Then the generating function  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  satisfies the asymptotic estimate

$$f(z) \sim \frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}} \quad (z \rightarrow 1^-).$$

This estimate remains valid when  $z$  tends to 1 in any sector with vertex at 1 symmetric about the horizontal axis, and with opening angle less than  $\pi$ .

To show that  $\mathcal{A}_{\mathbb{F}}$  is in fact a field, we need the following lemma. It indicates that depending on whether  $\mathbb{F}$  is a real field or not, every real algebraic number or every algebraic number can appear as a limit.

**Lemma 6.** Let  $p(z) \in \mathbb{F}[z]$  be an irreducible polynomial of degree  $d$ . Then there is a square-free polynomial  $P(z, y) \in \mathbb{F}[z, y]$  of degree  $d$  in  $y$  and admitting  $d$  distinct analytic algebraic functions  $f(z) \in \mathbb{F}[[z]]$  with  $P(z, f(z)) = 0$  in a neighborhood of 0 such that 1 is the only dominant singularity of each  $f$  and

1. if  $\mathbb{F} \subseteq \mathbb{R}$ , then for each root  $\xi \in \bar{\mathbb{F}} \cap \mathbb{R}$  of  $p(z)$  there exists a solution  $f(z)$  of  $P(z, y)$  with  $\lim_{n \rightarrow \infty} [z^n]f(z) = \xi$ ;
2. if  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ , then for each root  $\xi \in \bar{\mathbb{F}}$  of  $p(z)$  there exists a solution  $f(z)$  of  $P(z, y)$  with  $\lim_{n \rightarrow \infty} [z^n]f(z) = \xi$ .

*Proof.* The two assertions can be proved simultaneously as follows.

Let  $\varepsilon > 0$  be such that any two (real or complex) roots of  $p$  have a distance of more than  $\varepsilon$  to each other. Such an  $\varepsilon$  exists because  $p$  is a polynomial, and polynomials have only finitely many roots. The roots of a polynomial depend continuously on its coefficients. Therefore there exists a real number  $\delta > 0$  so that perturbing the coefficients by up to  $\delta$  won't perturb the roots by more than  $\varepsilon/2$ . Any positive smaller number than  $\delta$  will have the same property.

By the choice of  $\varepsilon$ , any such perturbation of the polynomial will have exactly one (real or complex) root in each of the balls of radius  $\varepsilon/2$  entered at the roots of  $p$ .

Let  $\xi$  be a root of  $p$ . If  $\xi = 0$ , then  $p(z) = z$ . Letting  $P(z, y) = y$  yields the assertions.

Assume that  $\xi \neq 0$ . Let  $m \in \mathbb{F}$  be the maximal modulus of coefficients of  $p$ . Then  $m \neq 0$  since  $p$  is irreducible. Therefore, we always can find a number  $a_0 \in \mathbb{F}$  such that  $|a_0 - \xi| < \delta/m$ , with the  $\delta$  from above. Indeed, we have the following case distinction.

For part **1** where  $\mathbb{F} \subseteq \mathbb{R}$ , we only consider  $\xi \in \overline{\mathbb{F}} \cap \mathbb{R}$ . In this case,  $\mathbb{F}$  is dense in  $\mathbb{R}$  since  $\mathbb{F} \supseteq \mathbb{Q}$ . Hence such  $a_0 \in \mathbb{F} \subseteq \mathbb{R}$  exists.

For part **2** where  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ , there exists a non-real complex number  $\alpha$  in  $\mathbb{F}$ . Therefore,  $\mathbb{Q}(\alpha)$  is dense in  $\mathbb{C}$ . Since  $\mathbb{Q}(\alpha) \subseteq \mathbb{F}$ , such  $a_0 \in \mathbb{F}$  is guaranteed by the density of  $\mathbb{F}$  in  $\mathbb{C}$ .

After finding  $a_0 \in \mathbb{F}$  with  $|a_0 - \xi| < \delta/m$ , for both cases, we have

$$|p(a_0)| = |p(a_0) - p(\xi)| \leq m|a_0 - \xi| < \delta.$$

Replace this  $\delta$  by  $|p(a_0)|$  for such a choice of  $a_0$ . The remaining argument works for both cases.

Consider the perturbation  $\tilde{p}(y) = p(y) - p(a_0)(1 - z)$ . For any  $z \in [0, 1]$  we have

$$|-p(a_0)(1 - z)| < |p(a_0)| = \delta.$$

Therefore, as  $z$  moves from 0 to 1, each root of  $p(y) - p(a_0)$  moves to the corresponding root of  $p(y)$ , which belongs to the same ball. In particular, the root  $a_0$  of  $\tilde{p}|_{z=0}$  will move to the root  $\xi$  of  $\tilde{p}|_{z=1}$ . Define

$$P(z, y) = p((1 - z)y) - p(a_0)(1 - z) \in \mathbb{F}[z, y].$$

We claim that  $P(z, y)$  determines an analytic algebraic function  $f(z)$  in  $\mathbb{F}[[z]]$  with the dominant singularity 1 and whose coefficient sequence converges to  $\xi$ . To prove this, we make an ansatz

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the  $a_0$  is from above and  $(a_n)_{n=1}^{\infty}$  are to be determined. Observe that for any  $c(z) \in \mathbb{F}[[z]]$ ,  $c(z)/(1 - z)$  is a root of  $P(z, y)$  if and only if  $c(z)$  is a root of  $\tilde{p}(y)$ , so  $f(z)$  admits the following Laurent expansion at  $z = 1$ ,

$$f(z) = \frac{\xi}{1 - z} + \sum_{n=0}^{\infty} b_n (1 - z)^n \quad \text{for } b_n \in \mathbb{C}.$$

Hence  $z = 1$  is a singularity of  $f(z)$  as  $\xi \neq 0$ .

The above argument also implies that  $z = 1$  is the only dominant singularity of  $f(z)$ . Indeed, note that  $z = 1$  is the only root of the leading coefficient of  $P(z, y)$  w.r.t.  $y$ , so the other singularities of  $f(z)$  could only be branch points, i.e., roots of discriminant of  $P(z, y)$  w.r.t.  $y$ . However, the choices of  $\varepsilon$  and  $\delta$  make it impossible for  $f(z)$  to have branch points in the disk  $|z| \leq 1$ , because in order to have a branch point, two roots of the polynomial  $P(z, y)$  w.r.t.  $y$  would need to touch each other, and we have ensured that they are always separated by more than  $\varepsilon$ . Consequently,  $z = 1$  is the dominant singularity of  $f(z)$ , which gives  $a_n \sim \xi$  as  $n \rightarrow \infty$  by part **1** of Theorem **5**. Therefore we have  $\lim_{n \rightarrow \infty} a_n = \xi$  since  $\xi \neq 0$ .

To complete the proof, it remains to show that the coefficients of  $f(z)$  are indeed in  $\mathbb{F}$ . This is observed by plugging the ansatz of  $f(z)$  into  $P(z, y)$  and comparing the coefficients of like powers of  $z$  to zero. Since  $p(z)$  is irreducible and  $\xi$  is arbitrary, one sees that  $P(z, y)$  admits  $d$  distinct analytic solutions in  $\mathbb{F}[[z]]$  in a neighborhood of 0.  $\square$

The following theorem clarifies the converse direction for algebraic sequences. It turns out that every element in  $\mathcal{A}_{\mathbb{F}}$  is algebraic over  $\mathbb{F}$ .

**Theorem 7.** *Let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$ .*

1. *If  $\mathbb{F} \subseteq \mathbb{R}$ , then  $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}} \cap \mathbb{R}$ .*
2. *If  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ , then  $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}}$ .*

*Proof.* 1. Let  $\xi \in \bar{\mathbb{F}} \cap \mathbb{R}$ . Then there is an irreducible polynomial  $p(z) \in \mathbb{F}[z]$  such that  $p(\xi) = 0$ . By part 1 of Lemma 6,  $\xi$  is in fact a limit of an algebraic sequence in  $\mathbb{F}^{\mathbb{N}}$ , which implies  $\xi \in \mathcal{A}_{\mathbb{F}}$ .

To show the converse inclusion, we let  $\xi \in \mathcal{A}_{\mathbb{F}}$ . When  $\xi = 0$ , there is nothing to show. Assume that  $\xi \neq 0$ . Then there is an algebraic sequence  $(a_n)_{n=0}^{\infty} \in \mathbb{F}^{\mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} a_n = \xi$ . Since  $\xi \neq 0$ , we have  $a_n \sim \xi$  ( $n \rightarrow \infty$ ).

Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . Clearly  $f(z)$  is an algebraic function over  $\mathbb{F}$ . By part 2 of Theorem 5,  $f(z) \sim \xi/(1-z)$  ( $z \rightarrow 1^-$ ), implying that  $z = 1$  is a simple pole of  $f(z)$  and

$$f(z) = \frac{\xi}{1-z} + \sum_{n=0}^{\infty} b_n (1-z)^n \quad \text{for } (b_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}.$$

Setting  $g(z) = f(z)(1-z)$  establishes that

$$g(z) = \xi + \sum_{n=0}^{\infty} b_n (1-z)^{n+1},$$

and then  $g(z)$  is analytic at 1. Sending  $z$  to 1 gives  $g(1) = \xi$ . By closure properties,  $g(z)$  is again an algebraic function over  $\mathbb{F}$ . Thus  $\xi = g(1) \in \bar{\mathbb{F}} \cap \mathbb{R}$  as  $\mathbb{F} \subseteq \mathbb{R}$ .

2. By part 2 of Lemma 6 and a similar argument as above, we have  $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}}$ . □

If we were to consider the class  $\mathcal{C}_{\mathbb{F}}$  of limits of convergent sequences in  $\mathbb{F}$  satisfying linear recurrence equations with constant coefficients over  $\mathbb{F}$ , sometimes called C-finite sequences, then an argument analogous to the above proof would imply that  $\mathcal{C}_{\mathbb{F}} \subseteq \mathbb{F}$ , because the power series corresponding to such sequences are rational functions, and the values of rational functions over  $\mathbb{F}$  at points in  $\mathbb{F}$  evidently gives values in  $\mathbb{F}$ . The converse direction  $\mathbb{F} \subseteq \mathcal{C}_{\mathbb{F}}$  is trivial, so we have  $\mathcal{C}_{\mathbb{F}} = \mathbb{F}$ .

**Corollary 8.** *If  $\mathbb{F} \subseteq \mathbb{R}$ , then  $\bar{\mathbb{F}} = \mathcal{A}_{\mathbb{F}(i)} = \mathcal{A}_{\mathbb{F}}[i] = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}}$ , where  $i$  is the imaginary unit.*

*Proof.* Since  $\mathcal{A}_{\mathbb{F}}$  is a ring and  $i^2 = -1 \in \mathbb{F} \subseteq \mathcal{A}_{\mathbb{F}}$ , we have  $\mathcal{A}_{\mathbb{F}}[i] = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}}$ . Since  $i \in \bar{\mathbb{F}}$  and  $\mathbb{F} \subseteq \mathbb{R}$ ,  $\bar{\mathbb{F}}$  is closed under complex conjugation and then

$$\bar{\mathbb{F}} = (\bar{\mathbb{F}} \cap \mathbb{R}) + i(\bar{\mathbb{F}} \cap \mathbb{R}) = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}},$$

by part 1 of Theorem 7. It follows from part 2 of Theorem 7 that  $\mathcal{A}_{\mathbb{F}(i)} = \overline{\mathbb{F}(i)}$ . Since  $\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{A}_{\mathbb{F}(i)}$  and  $i \in \mathcal{A}_{\mathbb{F}(i)}$ ,

$$\bar{\mathbb{F}} = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{A}_{\mathbb{F}(i)} = \overline{\mathbb{F}(i)} = \bar{\mathbb{F}},$$

The assertion holds. □

The following lemma says that every element in  $\overline{\mathbb{F}}$  can be written as the value at 1 of an analytic algebraic function vanishing at zero, provided that  $\mathbb{F}$  is dense in  $\mathbb{C}$ . This will be used in the next section to extend the evaluation domain.

**Lemma 9.** *Let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$  with  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ . Let  $p(z) \in \mathbb{F}[z]$  be an irreducible polynomial of degree  $d$ . Assume that  $\xi_1, \dots, \xi_d$  are all the (distinct) roots of  $p$  in  $\overline{\mathbb{F}}$ . Then there is a square-free polynomial  $P(z, y) \in \mathbb{F}[z, y]$  of degree  $d$  in  $y$  and admitting  $d$  distinct analytic algebraic functions  $g_1(z), \dots, g_d(z) \in \mathbb{F}[[z]]$  with  $P(z, g_j(z)) = 0$  in a neighborhood of 0 such that all  $g_j$ 's are analytic in the disk  $|z| \leq 1$  with  $g_j(0) = 0$  and, after reordering (if necessary),  $g_j(1) = \xi_j$ .*

*Proof.* By part 2 of Lemma 6, there exists a square-free polynomial  $\tilde{P}(z, y) \in \mathbb{F}[z, y]$  of degree  $d$  in  $y$  and admitting  $d$  distinct analytic algebraic functions  $f_1(z), \dots, f_d(z) \in \mathbb{F}[[z]]$  with  $P(z, f_j(z)) = 0$  in a neighborhood of 0 such that 1 is the only dominant singularity of each  $f_j(z)$  and, after reordering (if necessary),

$$\lim_{n \rightarrow \infty} [z^n] f_j(z) = \xi_j, \quad j = 1, \dots, d.$$

If  $\xi_j = 0$  for some  $j$  then  $p(z) = z$ . Letting  $P(z, y) = y$  yields the assertion. Otherwise all roots  $\xi_1, \dots, \xi_d$  are nonzero, and thus  $[z^n] f_j(z) \sim \xi_j$  ( $n \rightarrow \infty$ ) for  $j = 1, \dots, d$ . By part 2 of Theorem 5,  $f_j(z) \sim \xi_j / (1 - z)$  ( $z \rightarrow 1^-$ ), which implies that  $z = 1$  is a simple pole of each  $f_j(z)$ . Let  $g_j(z) = f_j(z)z(1 - z)$  for each  $j$ . Then  $g_1(z), \dots, g_d(z)$  are distinct and each  $g_j(z) \in \mathbb{F}[[z]]$  is analytic in the disk  $|z| \leq 1$  with  $g_j(0) = 0$  and  $g_j(1) = \xi_j$ . By closure properties,  $g_j(z)$  is again an algebraic function over  $\mathbb{F}$ . Define a square-free polynomial

$$P(z, y) = \prod_{j=1}^d (y - g_j(z)) = \prod_{j=1}^d (y - f_j(z)z(1 - z)) \in \overline{\mathbb{F}(z)}[y].$$

Since  $P(z, y)$  is symmetric in  $f_1, \dots, f_d$ , the polynomial  $P(z, y)$  is in  $\mathbb{F}[z, y]$ . The lemma follows.  $\square$

## 4 D-finite Numbers

Let us now return to the study of D-finite numbers. Let  $R$  be a subring of  $\mathbb{C}$  and  $\mathbb{F}$  be a subfield of  $\mathbb{C}$ . Recall that by Definition 1, the elements of  $\mathcal{D}_{R, \mathbb{F}}$  are exactly limits of convergent sequences in  $R^{\mathbb{N}}$  which are P-recursive over  $\mathbb{F}$ . Some facts about P-recursive sequences translate directly into facts about  $\mathcal{D}_{R, \mathbb{F}}$ .

**Proposition 10.**

1.  $R \subseteq \mathcal{D}_{R, \mathbb{F}}$  and  $\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{D}_{\mathbb{F}}$ .
2. If  $R_1 \subseteq R_2$  then  $\mathcal{D}_{R_1, \mathbb{F}} \subseteq \mathcal{D}_{R_2, \mathbb{F}}$ , and if  $\mathbb{F} \subseteq \mathbb{E}$  then  $\mathcal{D}_{R, \mathbb{F}} \subseteq \mathcal{D}_{R, \mathbb{E}}$ .
3.  $\mathcal{D}_{R, \mathbb{F}}$  is a subring of  $\mathbb{C}$ . Moreover, if  $R$  is an  $\mathbb{F}$ -algebra then so is  $\mathcal{D}_{R, \mathbb{F}}$ .
4. If  $\mathbb{E}$  is an algebraic extension field of  $\mathbb{F}$ , then  $\mathcal{D}_{R, \mathbb{F}} = \mathcal{D}_{R, \mathbb{E}}$ .
5. If  $R \subseteq \mathbb{F}$ , then  $\mathcal{D}_{R, \mathbb{F}} = \mathcal{D}_{R, \text{Quot}(R)}$ .

6. If  $R$  and  $\mathbb{F}$  are closed under complex conjugation, then so is  $\mathcal{D}_{R,\mathbb{F}}$ .

In this case, we have  $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$ .

Moreover, if the imaginary unit  $i \in \mathcal{D}_{R,\mathbb{F}}$  then  $\mathcal{D}_{R,\mathbb{F}} = \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}} + i\mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$ .

*Proof.* 1. The first inclusion is clear because every element of  $R$  is the limit of a constant sequence, and every constant sequence is P-recursive. The second inclusion follows from the fact that algebraic functions are D-finite, and the coefficient sequences of D-finite functions are P-recursive.

2. Clear.

3. Follows directly from the corresponding closure properties for P-recursive sequences.

4. Follows directly from part 1 of Lemma 4.

5. Follows directly from part 2 of Lemma 4.

6. For any convergent sequence  $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ , we have

$$\operatorname{Re} \left( \lim_{n \rightarrow \infty} a_n \right) = \lim_{n \rightarrow \infty} \operatorname{Re}(a_n), \quad \operatorname{Im} \left( \lim_{n \rightarrow \infty} a_n \right) = \lim_{n \rightarrow \infty} \operatorname{Im}(a_n) \quad \text{and thus} \quad \overline{\lim_{n \rightarrow \infty} a_n} = \lim_{n \rightarrow \infty} \bar{a}_n.$$

Hence the first assertion follows by  $(\bar{a}_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$  and part 3 of Lemma 4.

Since  $R$  is closed under complex conjugation,  $(\operatorname{Re}(a_n))_{n=0}^{\infty} \in (R \cap \mathbb{R})^{\mathbb{N}}$ . Then the inclusion  $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} \subseteq \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$  can be shown similarly as the first assertion. The converse direction holds by part 2. Therefore  $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$ .

If  $i \in \mathcal{D}_{R,\mathbb{F}}$ , then  $\mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}} + i\mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}} \subseteq \mathcal{D}_{R,\mathbb{F}}$  since  $\mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}} \subseteq \mathcal{D}_{R,\mathbb{F}}$ . To show the converse inclusion, let  $\xi \in \mathcal{D}_{R,\mathbb{F}}$ . Then  $\bar{\xi} \in \mathcal{D}_{R,\mathbb{F}}$  by the first assertion. Since  $i \in \mathcal{D}_{R,\mathbb{F}}$  and  $R$  is closed under complex conjugation,  $\operatorname{Re}(\xi), \operatorname{Im}(\xi)$  both belong to  $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$  by the second assertion. Therefore  $\xi = \operatorname{Re}(\xi) + i\operatorname{Im}(\xi) \in \mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}} + i\mathcal{D}_{R \cap \mathbb{R}, \mathbb{F}}$ . □

### Example 11.

1. We have  $\mathcal{D}_{\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\pi, \sqrt{2})} = \mathcal{D}_{\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{2})} = \mathcal{D}_{\mathbb{Q}(\sqrt{2}), \mathbb{Q}}$ . The first identity holds by part 5, the second by part 4 of the proposition.

2. We have  $\mathcal{D}_{\bar{\mathbb{Q}}, \mathbb{Q}} = \mathcal{D}_{\bar{\mathbb{Q}}, \mathbb{R}}$ . The inclusion “ $\subseteq$ ” is clear by part 2. For the inclusion “ $\supseteq$ ”, let  $\xi \in \mathcal{D}_{\bar{\mathbb{Q}}, \mathbb{R}}$ . Then  $\xi = a + ib$  for some  $a, b \in \mathbb{R}$ , and there exists a sequence  $(a_n + ib_n)_{n=0}^{\infty}$  in  $\bar{\mathbb{Q}}^{\mathbb{N}}$  and an operator  $L \in \mathbb{R}[n]\langle S_n \rangle$  such that  $L \cdot (a_n + ib_n) = 0$  and  $\lim_{n \rightarrow \infty} (a_n + ib_n) = a + ib$ . Since the coefficients of  $L$  are real, we then have  $L \cdot a_n = 0$  and  $L \cdot b_n = 0$ . Furthermore,  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Therefore,

$$a, b \in \mathcal{D}_{\bar{\mathbb{Q}} \cap \mathbb{R}, \mathbb{R}} \stackrel{\text{part 5}}{=} \mathcal{D}_{\bar{\mathbb{Q}} \cap \mathbb{R}, \bar{\mathbb{Q}} \cap \mathbb{R}} \stackrel{\text{part 4}}{=} \mathcal{D}_{\bar{\mathbb{Q}} \cap \mathbb{R}, \mathbb{Q}},$$

whence  $a + ib \in \mathcal{D}_{\bar{\mathbb{Q}} \cap \mathbb{R}, \mathbb{Q}} + i\mathcal{D}_{\bar{\mathbb{Q}} \cap \mathbb{R}, \mathbb{Q}} \stackrel{\text{part 6}}{=} \mathcal{D}_{\bar{\mathbb{Q}}, \mathbb{Q}}$ , as claimed.

Lemma 9 motivates the following theorem, which says that every D-finite number is essentially the value of an analytic D-finite function at 1.

**Theorem 12.** *Let  $R$  be a subring of  $\mathbb{C}$  and let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$ . Then for every  $\xi \in \mathcal{D}_{R,\mathbb{F}}$ , there exists  $g(z) \in R[[z]]$  D-finite over  $\mathbb{F}$  and analytic at 1 such that  $\xi = g(1)$ .*

*Proof.* The statement is clear when  $\xi = 0$ . Assume that  $\xi$  is nonzero. Then there exists a sequence  $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ , P-recursive over  $\mathbb{F}$ , such that  $\lim_{n \rightarrow \infty} a_n = \xi$ . Since  $\xi \neq 0$ , we have  $a_n \sim \xi$  ( $n \rightarrow \infty$ ). Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . By Theorem 5,  $f(z) \sim \xi/(1-z)$  as  $z \rightarrow 1^-$ , which implies that  $z = 1$  is a simple pole of  $f(z)$ . Let  $g(z) = f(z)(1-z)$ . Then  $g(z)$  belongs to  $R[[z]]$  and is analytic at  $z = 1$ . Write

$$f(z) = \frac{\xi}{1-z} + \sum_{n=0}^{\infty} b_n (1-z)^n \quad \text{with } b_n \in \mathbb{C}.$$

Then we get

$$g(z) = f(z)(1-z) = \xi + \sum_{n=0}^{\infty} b_n (1-z)^{n+1},$$

which gives  $\xi = g(1)$ . The assertion follows by noticing that  $g(z)$  is D-finite over  $\mathbb{F}$  due to closure properties.  $\square$

**Example 13.** *We have  $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = g(1)$  for  $g(z) = \sum_{n=1}^{\infty} \frac{1}{n^3} z^n \in \mathbb{Q}[[z]]$ . Note that  $g$  is D-finite and analytic at 1.*

Note that the above theorem implies that D-finite numbers are computable when the ring  $R$  and the field  $\mathbb{F}$  consist of computable numbers. This allows the construction of (artificial) numbers that are not D-finite.

Some kind of converse of Theorem 12 can be proved for the case when  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ . Note that this condition is equivalent to saying that  $\mathbb{F}$  is dense in  $\mathbb{C}$ . To this end, we first need to develop several lemmas.

The following lemma says that the value of a D-finite function at any non-singular point in  $\bar{\mathbb{F}}$  can be represented by the value of another D-finite function at 1.

**Lemma 14.** *Let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$  with  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$  and  $R$  be a subring of  $\mathbb{C}$  containing  $\mathbb{F}$ . Assume that  $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$  is analytic and annihilated by a nonzero operator  $L \in \mathbb{F}[z]\langle D_z \rangle$  with zero an ordinary point. Then for any non-singular point  $\zeta \in \bar{\mathbb{F}}$  of  $L$ , there exists an analytic function  $h(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$  and a nonzero operator  $M \in \mathbb{F}[z]\langle D_z \rangle$  with 0 and 1 ordinary points such that  $M \cdot h(z) = 0$  and  $f(\zeta) = h(1)$ .*

*Proof.* Let  $\zeta \in \bar{\mathbb{F}}$  be a non-singular point of  $L$ . Then there exists an irreducible polynomial  $p(z) \in \mathbb{F}[z]$  such that  $p(\zeta) = 0$ . Let  $\zeta_1 = \zeta, \dots, \zeta_d$  be all the roots of  $p$  in  $\bar{\mathbb{F}}$ . By Lemma 9, there exists a square-free polynomial  $P(z, y) \in \mathbb{F}[z, y]$  of degree  $d$  in  $y$  and admitting  $d$  distinct analytic algebraic functions  $g_1(z), \dots, g_d(z) \in \mathbb{F}[[z]]$  with  $P(z, g_j(z)) = 0$  in a neighborhood of 0. Moreover,  $g_1(z), \dots, g_d(z)$  are all analytic in the disk  $|z| \leq 1$  with  $g_j(1) = \zeta_j$  and  $g_j(0) = 0$ .

Since  $g_1(1) = \zeta$  is not a singularity of  $L$  by assumption, none of  $g_j(1) = \zeta_j$  is a singularity of  $L$ . Suppose otherwise that for some  $2 \leq \ell \leq d$ , the point  $g_\ell(1) = \zeta_\ell$  is a root of  $\text{lc}(L)$ . Since  $\text{lc}(L) \in \mathbb{F}[z]$  and  $p$  is the minimal polynomial of  $\zeta_\ell$  over  $\mathbb{F}$ , we know that  $p$  divides  $\text{lc}(L)$  over  $\mathbb{F}$ . Thus  $\zeta$  is also a root of  $\text{lc}(L)$ , a contradiction.

Note that  $g_1(z), \dots, g_d(z)$  are analytic in  $|z| \leq 1$  and  $g_j(0) = 0$ . By Theorem 3, there exists a nonzero operator  $M \in \mathbb{F}[z]\langle D_z \rangle$  with  $M \cdot (f \circ g_1) = 0$  which does not have 0 or 1 among its singularities. By part 1 of Proposition 10,  $\mathbb{F} \subseteq R \subseteq \mathcal{D}_{R,\mathbb{F}}$ . Since  $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$  and  $g_1(z) \in \mathbb{F}[[z]]$  with  $g_1(0) = 0$ , we have  $f(g_1(z)) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ . Setting  $h(z) = f(g_1(z))$  completes the proof.  $\square$

With the above lemma, it suffices to consider the case when the evaluation point is in  $R \cap \mathbb{F}$ . This is exactly what the next two lemmas are concerned about.

**Lemma 15.** *Assume that  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R[[z]]$  is D-finite over  $\mathbb{F}$  and convergent in some neighborhood of 0. Let  $\zeta \in R \cap \mathbb{F}$  be in the disk of convergence. Then  $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$  for all  $k \in \mathbb{N}$ .*

*Proof.* For  $k \in \mathbb{N}$ , it is well-known that  $f^{(k)}(z) \in R[[z]]$  is also D-finite and has the same radius of convergence at zero as  $f(z)$ . Note that since  $f(z)$  is D-finite over  $\mathbb{F}$ , so is  $f^{(k)}(z)$ . Thus to prove the lemma, it suffices to show the case when  $k = 0$ , i.e.,  $f(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$ .

Since  $f(z)$  is D-finite over  $\mathbb{F}$ , the coefficient sequence  $(a_n)_{n=0}^{\infty}$  is P-recursive over  $\mathbb{F}$ . Note that  $\zeta \in R \cap \mathbb{F}$  is in the disk of convergence of  $f(z)$  at zero, so

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \lim_{n \rightarrow \infty} \sum_{\ell=0}^n a_\ell \zeta^\ell.$$

Since  $(\zeta^n)_{n=0}^{\infty}$  is P-recursive over  $\mathbb{F}$ , the assertion follows by noticing that  $(\sum_{\ell=0}^n a_\ell \zeta^\ell)_{n=0}^{\infty} \in R^{\mathbb{N}}$  is P-recursive over  $\mathbb{F}$  due to closure properties.  $\square$

**Example 16.** *Since  $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \in \mathbb{Q}[[z]]$  is D-finite over  $\mathbb{Q}$ , and converges everywhere, we get from the lemma that the numbers  $e, 1/e, \sqrt{e}$  belong to  $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}$ . More precisely, since we are currently only considering non-real fields  $\mathbb{F}$ , we could say that  $\exp(z)$  is D-finite over  $\bar{\mathbb{Q}}$ , therefore  $e, 1/e, \sqrt{e} \in \mathcal{D}_{\mathbb{Q},\bar{\mathbb{Q}}}$ , but by Proposition 10,  $\mathcal{D}_{\mathbb{Q},\bar{\mathbb{Q}}} = \mathcal{D}_{\mathbb{Q},\mathbb{Q}}$ .*

**Lemma 17.** *Let  $R$  be a subring of  $\mathbb{C}$  containing  $\mathbb{F}$ . Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_{R,\mathbb{F}}[[z]]$  be an analytic function. Assume that there exists a nonzero operator  $L \in \mathbb{F}[z]\langle D_z \rangle$  with zero an ordinary point such that  $L \cdot f(z) = 0$ . Let  $r > 0$  be the smallest modulus of roots of  $\text{lc}(L)$  and let  $\zeta \in \mathbb{F}$  with  $|\zeta| < r$ . Then  $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$  for all  $k \in \mathbb{N}$ .*

*Proof.* Let  $\rho$  be the order of  $L$ . Since zero is an ordinary point of  $L$ , there exist P-recursive sequences  $(c_n^{(0)})_{n=0}^{\infty}, \dots, (c_n^{(\rho-1)})_{n=0}^{\infty}$  in  $\mathbb{F}^{\mathbb{N}} \subseteq R^{\mathbb{N}}$  with  $c_j^{(m)}$  equal to the Kronecker delta  $\delta_{mj}$  for  $m, j = 0, \dots, \rho - 1$ , so that the set  $\{\sum_{n=0}^{\infty} c_n^{(m)} z^n\}_{m=0}^{\rho-1}$  forms a basis of the solution space of  $L$  near zero. Note that the singularities of solutions of  $L$  can only be roots of  $\text{lc}(L)$ . Hence the power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  as well as  $\sum_{n=0}^{\infty} c_n^{(m)} z^n$  for  $m = 0, \dots, \rho - 1$  are convergent in the disk  $|z| < r$ . It follows from  $|\zeta| < r$  and Lemma 15 that the set  $\{\sum_{n=0}^{\infty} c_n^{(m)} \zeta^n\}_{m=0}^{\rho-1}$  belongs to  $\mathcal{D}_{R,\mathbb{F}}$ . Since  $a_0, \dots, a_{\rho-1} \in \mathcal{D}_{R,\mathbb{F}}$ ,

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = a_0 \sum_{n=0}^{\infty} c_n^{(0)} \zeta^n + \dots + a_{\rho-1} \sum_{n=0}^{\infty} c_n^{(\rho-1)} \zeta^n$$

is D-finite by closure properties. In the same vein, we find that for  $k > 0$ , the derivative  $f^{(k)}(\zeta)$  also belongs to  $\mathcal{D}_{R,\mathbb{F}}$ .  $\square$

**Example 18.**

1. We know from Proposition 10 that  $\sqrt{2} \in \mathcal{D}_{\mathbb{Q}}$ . The series

$$(z+1)^{\sqrt{2}} = 1 + \sqrt{2}z + \left(1 - \frac{1}{\sqrt{2}}\right)z^2 + \dots \in \mathbb{Q}(\sqrt{2})[[z]] \subseteq \mathcal{D}_{\mathbb{Q}}[[z]]$$

is D-finite over  $\mathbb{Q}$ , an annihilating operator is  $(z+1)^2 D_z^2 + (z+1)D_z - 2$ . Here we have the radius  $r = 1$ . Taking  $\zeta = \sqrt{2} - 1$ , the lemma implies  $\sqrt{2}^{\sqrt{2}} \in \mathcal{D}_{\mathbb{Q}}$ .

2. Observe that the lemma refers to the singularities of the operator rather than to the singularities of the particular solution at hand. For example, it does not imply that  $J_1(1) \in \mathcal{D}_{\mathbb{Q},\mathbb{Q}}$ , where  $J_1(z)$  is the first Bessel function, because its annihilating operator is  $z^2 D_z^2 + z D_z + (z^2 - 1)$ , which has a singularity at 0. It is not sufficient that the particular solution  $J_1(z) \in \mathbb{Q}[[z]]$  is analytic at 0. Of course, in this particular example we see from the series representation  $J_1(1) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1/4)^n}{(n+1)n!^2}$  that the value belongs to  $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}$ .
3. The hypergeometric function  $f(z) := {}_2F_1(\frac{1}{3}, \frac{1}{2}, 1, z + \frac{1}{2})$  can be viewed as an element of  $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}[[z]]$ :

$$f(z) = \underbrace{\sqrt[3]{2} \sum_{n=0}^{\infty} \frac{(1/3)_n (1/2)_n}{n!^2} (-1)^n}_{\in \mathcal{D}_{\mathbb{Q}}} + \underbrace{\frac{\sqrt[3]{2}}{3} \sum_{n=0}^{\infty} \frac{(1/2)_n (4/3)_n}{(2)_n n!} (-1)^n z}_{\in \mathcal{D}_{\mathbb{Q}}} + \underbrace{\frac{2\sqrt[3]{2}}{3} \sum_{n=0}^{\infty} \frac{(1/2)_n (7/3)_n}{(3)_n n!} (-1)^n z^2 + \dots}_{\in \mathcal{D}_{\mathbb{Q}}}.$$

The function  $f$  is annihilated by the operator

$$L = 3(2z - 1)(2z + 1)D_z^2 + (22z - 1)D_z + 2.$$

This operator has a singularity at  $z = 1/2$ , and there is no annihilating operator of  $f$  which does not have a singularity there. Although  $f(1/2) = \frac{\Gamma(1/6)}{\Gamma(1/2)\Gamma(2/3)}$  is a finite and specific value, the lemma does not imply that this value is a  $D$ -finite number.

**Theorem 19.** Let  $\mathbb{F}$  be a subfield of  $\mathbb{C}$  with  $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$  and let  $R$  be a subring of  $\mathbb{C}$  containing  $\mathbb{F}$ . Assume that  $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$  is analytic and there exists a nonzero operator  $L \in \mathbb{F}[z]\langle D_z \rangle$  with zero an ordinary point such that  $L \cdot f(z) = 0$ . Further assume that  $\zeta \in \overline{\mathbb{F}}$  is not a singularity of  $L$ . Then  $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$  for all  $k \in \mathbb{N}$ .

*Proof.* By Lemma 14, it suffices to show the assertion holds for  $\zeta = 1$  (or more generally  $\zeta \in \mathbb{F}$ ). Now assume that  $\zeta \in \mathbb{F}$ . We apply the method of analytic continuation.

Let  $\mathcal{P}$  be a simple path with a finite cover  $\bigcup_{j=0}^s \mathcal{B}_{r_j}(\beta_j)$ , where  $s \in \mathbb{N}$ ,  $\beta_0 = 0$ ,  $\beta_s = \zeta$ ,  $\beta_j \in \mathbb{F}$ ,  $r_j > 0$  is the distance between  $\beta_j$  and the zero set of  $\text{lc}(L)$ , and  $\mathcal{B}_{r_j}(\beta_j)$  is the open circle centered at  $\beta_j$  and with radius  $r_j$ . Moreover,  $\beta_{j+1} \in \mathcal{B}_{r_j}(\beta_j)$  for each  $j$  (as illustrated by Figure 1). Such a path exists because  $\mathbb{F}$  is dense in  $\mathbb{C}$  and the zero set of  $\text{lc}(L)$  is finite. Since the  $\mathcal{P}$  avoids all roots of  $\text{lc}(L)$ , the function  $f(z)$  is analytic along  $\mathcal{P}$ . We next use induction on the index  $j$  to show that  $f^{(k)}(\beta_j) \in \mathcal{D}_{R,\mathbb{F}}$  for all  $k \in \mathbb{N}$ .

It is trivial when  $j = 0$  as  $f^{(k)}(\beta_0) = f^{(k)}(0) \in \mathcal{D}_{R,\mathbb{F}}$  for  $k \in \mathbb{N}$  by assumption. Assume now that  $0 < j \leq s$  and  $f^{(k)}(\beta_{j-1}) \in \mathcal{D}_{R,\mathbb{F}}$  for all  $k \in \mathbb{N}$ . We consider  $f(\beta_j)$  and its derivatives.

Recall that  $r_{j-1} > 0$  is the distance between  $\beta_{j-1}$  and the zero set of  $\text{lc}(L)$ . Since  $f(z)$  is analytic at  $\beta_{j-1}$ , it is representable by a convergent power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\beta_{j-1})}{n!} (z - \beta_{j-1})^n \quad \text{for all } |z - \beta_{j-1}| < r_{j-1}.$$

By the induction hypothesis,  $f^{(n)}(\beta_{j-1})/n! \in \mathcal{D}_{R,\mathbb{F}}$  for all  $n \in \mathbb{N}$  and thus  $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z - \beta_{j-1}]]$ . Let  $Z = z - \beta_{j-1}$ , i.e.,  $z = Z + \beta_{j-1}$ . Define  $g(Z) = f(Z + \beta_{j-1})$  and  $\tilde{L}$  to be the operator

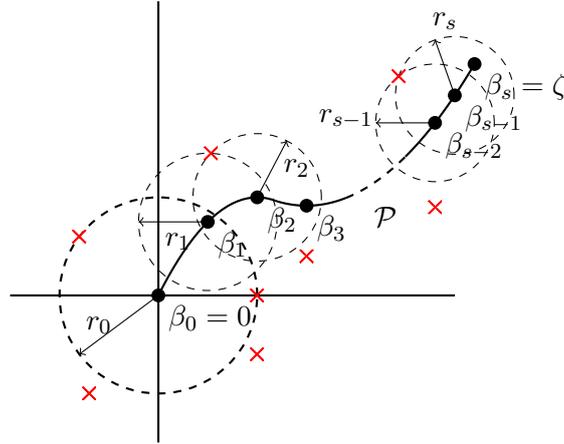


Figure 1: a simple path  $\mathcal{P}$  with a finite cover  $\bigcup_{j=0}^s \mathcal{B}_{r_j}(\beta_j)$  ( $\times$  stands for the roots of  $\text{lc}(L)$ )

obtained by replacing  $z$  in  $L$  by  $Z + \beta_j$ . Since  $\beta_{j-1} \in \mathbb{F} \subseteq \mathcal{D}_{R,\mathbb{F}}$  and  $D_z = D_Z$ , we have  $g(Z) \in \mathcal{D}_{R,\mathbb{F}}[[Z]]$  and  $\tilde{L} \in \mathbb{F}[Z]\langle D_Z \rangle$ . Note that  $L \cdot f(z) = 0$  and  $\beta_{j-1}$  is an ordinary point of  $L$  as  $r_{j-1} > 0$ . It follows that  $\tilde{L} \cdot g(Z) = 0$  and zero is an ordinary point of  $\tilde{L}$ . Moreover, we see that  $r_{j-1}$  is now the smallest modulus of roots of  $\text{lc}(\tilde{L})$ . Since  $|\beta_j - \beta_{j-1}| < r_{j-1}$ , applying Lemma 17 to  $g(Z)$  yields  $f^{(k)}(\beta_j) = g^{(k)}(\beta_j - \beta_{j-1}) \in \mathcal{D}_{R,\mathbb{F}}$  for  $k \in \mathbb{N}$ . Thus the assertion holds for  $j = s$ . The theorem follows.  $\square$

**Example 20.** *By the above theorem,  $\exp(\sqrt{2})$  and  $\log(1 + \sqrt{3})$  both belong to  $\mathcal{D}_{\mathbb{Q}}$ . We also have  $e^\pi \in \mathcal{D}_{\mathbb{Q}}$ . This is because  $e^\pi = (-1)^{-i}$  with  $i$  the imaginary unit, is equal to the value of the  $D$ -finite function  $(z + 1)^{-i} \in \mathbb{Q}(i)[[z]]$  at  $z = -2$  (outside the radius of convergence; analytically continued in consistency with the usual branch cut conventions) and then  $e^\pi \in \mathcal{D}_{\mathbb{Q}(i)} \cap \mathbb{R} = \mathcal{D}_{\mathbb{Q}}$ . Furthermore, as remarked in the introduction, the numbers obtained by evaluating a  $G$ -function at algebraic numbers which avoid the singularities of its annihilating operator are in  $\mathcal{D}_{\mathbb{Q}(i)}$ , because  $G$ -functions are  $D$ -finite.*

## 5 Open Questions

We have introduced the notion of  $D$ -finite numbers and made some first steps towards understanding their nature. We believe that, similarly as for  $D$ -finite functions, the class is interesting because it has good mathematical and computational properties and because it contains many special numbers that are of independent interest. We conclude this paper with some possible directions of future research.

**Evaluation at singularities.** While every singularity of a  $D$ -finite function must also be a singularity of its annihilating operator, the converse is in general not true. We have seen above that evaluating a  $D$ -finite function at a point which is not a singularity of its annihilating operator yields a  $D$ -finite number. It would be natural to wonder about the values of a  $D$ -finite function at singularities of its annihilating operator, including those at which the given function is not analytic but its evaluation is finite. Also, we always consider zero as an ordinary point of the annihilating operator. If this is not the case, the method used in the paper fails, as pointed out by part 2 of Example 18.

**Quotients of D-finite numbers.** The set of algebraic numbers forms a field, but we do not have a similar result for D-finite numbers. It is known that the set of D-finite functions does not form a field. Instead, Harris and Sibuya [9] showed that a D-finite function  $f$  admits a D-finite multiplicative inverse if and only if  $f'/f$  is algebraic. This explains for example why both  $e$  and  $1/e$  are D-finite, but it does not explain why both  $\pi$  and  $1/\pi$  are D-finite. It would be interesting to know more precisely under which circumstances the multiplicative inverse of a D-finite number is D-finite. Is  $1/\log(2)$  a D-finite number? Are there choices of  $R$  and  $\mathbb{F}$  for which  $\mathcal{D}_{R,\mathbb{F}}$  is a field?

**Roots of D-finite functions.** A similar pending analogy concerns compositional inverses. We know that if  $f$  is an algebraic function, then so is its compositional inverse  $f^{-1}$ . The analogous statement for D-finite functions is not true. Nevertheless, it could still be true that the values of compositional inverses of D-finite functions are D-finite numbers, although this seems somewhat unlikely. A particularly interesting special case is the question whether (or under which circumstances) the roots of a D-finite function are D-finite numbers.

**Evaluation at D-finite number arguments.** We see that the class  $\mathcal{C}_{\mathbb{F}}$  of limits of convergent C-finite sequences is the same as the values of rational functions at points in  $\mathbb{F}$ , namely the field  $\mathbb{F}$ . Similarly, the class  $\mathcal{A}_{\mathbb{F}}$  of limits of convergent algebraic sequences essentially consists of the values of algebraic functions at points in  $\overline{\mathbb{F}}$ . Continuing this pattern, is the value of a D-finite function at a D-finite number again a D-finite number? If so, this would imply that also numbers like  $e^{e^{e^e}}$  are D-finite. Since  $1/(1-z)$  is a D-finite function, it would also imply that D-finite numbers form a field.

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