Generalizing some Results in Field Theory for Rings

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Abstract

The main results of this paper is divided into two parts. In the first part we show some interesting results on Galois-like theory for commutative reduced Baer rings and give some interesting facts about the so-called *splitting rings* and their group of automorphisms. In the second part we concentrate on Artin-Schreier Theorem and we show a generalized form of this theorem for commutative reduced Baer normal rings. To show the results we recall important, but rather less known, results and definitions by Raphael: namely generalized *algebraic extensions* and *algebraic closures* for reduced rings (which is equivalent to the classical definition for fields). We give examples and show why essential extension plays an important role when considering algebraic extensions. This allows us to define a splitting ring of a polynomial over a Baer reduced ring and show that, like in fields, such rings are finitely generated modules over this Baer ring. We show, however, that these rings (unlike in fields) will not necessarily induce a finite group of automorphism and these group may not even be finitely generated. Nevertheless, we still show that, like in fields, this group will be a torsion group with finite exponent. Moving to generalized Artin-Schreier Theorem, we show that if the algebraic closure of a Baer reduced normal real ring is a finitely generated module (over the given Baer ring) then adjoining the real Baer ring with an imaginary element $\sqrt{-1}$ will give us the algebraic closure of the Baer ring. We show, by an example, why it is important to require that the base ring is Baer.

Keywords: Artin-Schreier Theorem, splitting rings, total integral closure, Galois theory

1 Introduction

In this section, we give some introductory material about commutative unitary reduced rings. Note that some of the results or definitions can be generalized to other categories (e.g. rings that are not reduced, modules etc.). If σ is any endomorphism (between the same object in a given category) and if $k \in \mathbb{N}$, then by σ^k we mean the endomorphism created by composing σ with itself k-times.

For a commutative ring A, $T(A)$ denotes the total quotient ring of A. Unless otherwise stated, all rings in this papers are commutative unitary and reduced. All ring homomorphisms are such that 1 is mapped to 1. Let A be a ring, then $\operatorname{Spec} A$ is the set of prime ideals of A endowed with the Zariski topology and $\operatorname{Min} A$ is the set of minimal prime ideals of A considered as a subspace of $\operatorname{Spec} A$. For a ring A and $a \in A$ the the set $D_A(a) \subset \operatorname{Spec} A$ is the basic open set

$$D_A(a) = \{\mathfrak{p} \in \operatorname{Spec} A \,:\, a \notin B\}$$

and $V_A(a)$ is the basic closed set

$$V_A(a) = \{\mathfrak{p} \in \operatorname{Spec} A \,:\, a \in B\}$$

Here are some results and more definitions

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- Let A be a subring of a ring B, then B is an *essential extension of* A (in the category of commutative rings) if for all $b \in B \backslash \{0\}$ there exists a $c \in B$ such that $bc \in A \backslash \{0\}$. There is a generalized definition for essential extensions in any category $\mathcal{C}$. Let $\mathcal{C}$ be a category, then a monomorphism $f : a \to b$ is said to be an *essential monomorphism* (or extension) provided that for all morphism/arrow $g : b \to c$ such that the composition $g \circ f : a \to c$ is a monomorphism, it follows that g is a monomorphism. Some authors (e.g. Hochster in [12]) also use *tight extension* to mean essential extension.

- Let A be a subring of a reduced ring B, then B is said to be a *rational extension* of A if for all $b \in B \backslash \{0\}$ there exists an $a \in A$ such that $ab \in A \backslash \{0\}$. This definition can be generalized for non-reduced rings but we will confine ourselves to reduced commutative unitary rings. The study of such ring extensions became quite popular in the 50's (notably by Utumi, Lambek, Findlay and Johnson).

- It is known (see [11]) that if A is reduced, commutative and unitary then there is a maximum rational extension $Q(A)$, i.e. a rational extension $Q(A)$ of A such that any other rational extension of A is A-isomorphic (i.e. isomorphism fixing A) to a subring of $Q(A)$. $Q(A)$ is called the *rational completion* or the *complete ring of quotients* of A. If $Q(A)$ is ring-isomorphic to A, then A is said to be *rationally complete.*

- As used by Raphael (see [9], [17] and [18]), we shall call an essential and integral extension of a commutative ring the *algebraic extension* of this ring. This definition coincides with the classical definition of algebraic extension when working with the category fields. It has been shown (see [3], [10] and [12]) that a commutative reduced unitary ring A have a maximum algebraic extension $\bar{A}$, in the sense that any algebraic extension of $\bar{A}$ is A-isomorphic to $\bar{A}$, which we shall call the *algebraic closure* of the ring A. When dealing with fields, this definition of algebraic closure also coincides with the classical definition of algebraic closure. Most authors also call the *algebraic closure* the *total integral closure* of the ring and in this paper we may sometimes use this name.

- A commutative ring A is said to be *real* iff for all $n \in \mathbb{N}$ the following holds

$$a_1^2 + \cdots + a_n^2 = 0 \Leftrightarrow a_1, \ldots, a_n = 0 \qquad \forall a_1, \ldots, a_n \in A$$

  Clearly, every real ring is reduced and the total quotient of a real ring is real as well (specifically, the quotient field of a real domain is real). A real ring A is said to be *real closed* iff there is no strict algebraic extension of A that is also real. By Zorn's lemma every real ring has an algebraic extension that is real closed. Real closed rings (using this definition) were first introduced by Sankaran and Varadarajan in [19]. It was then more extensively studied in the PhD thesis of Capco (who defined this originally as *real closed* $*$ to distinguish with other definition with similar name). Real closed rings are also Baer rings (see [5] Remark 28). Integral domains are real closed iff they are integrally closed in their quotient fields and their quotient fields are real closed fields (see [19] Proposition 2). Commutative von Neumann regular rings are real closed iff it is Baer and all the residue fields are real closed (see [5] Theorem 34).

The proposition below illustrates that one is able to arbitrarily strictly extend any reduced ring integrally (the proposition uses a field but this can be generalized). Thus, as discussed by Raphael [17], Borho [3], Enochs [10] and Hochster [12], it is necessary to involve *essential extensions* when defining algebraic extensions and algebraic closures of commutative reduced rings.

Proposition 1. Let K be a field and L be an algebraic extension of K then for any $n \in \mathbb{N}$ the ring L^n (componentwise addition and multiplication) is an integral extension of K.

Proof. There is a natural monomorphism from K to L^n (diagonal homomorphism) that brings each $k \in K$ to $(k, k, \ldots, k) \in L^n$. In this way we identify K as a subring of L^n, but to avoid confusion we write $\bar{k}$ (instead of $k \in K$) to denote $(k, k, \ldots, k)$. Furthermore, there is a canonical ring homomorphism $K[x] \to L^n[x]$, between polynomial rings, that maps each $\sum_{i=1}^m a_i x^i \in K[x]$ to $\sum_{i=1}^m \bar{a}_i x^i \in L^n[x]$. We abuse notation and denote the image of $f \in K[x]$ (by this homomorphism) by $\bar{f} \in L^n[x]$. Consider now $b_i \in L$ for $i = 1, \ldots, n$ (i.e. an arbitrary $(b_1, b_2, \ldots, b_n) \in L^n$). There exists non-constant monic polynomials $f_1, f_2, \ldots, f_n \in L$ such that b_i is a zero of f_i for $i = 1, \ldots, n$. So if we consider the polynomial $f := \prod_{i=1}^n f_i \in K[x]$, then the canonical image $\bar{f}$ in $L^n[x]$ has a zero $(b_1, b_2, \ldots, b_n)$. Thus, L^n is an integral extension of K. □

Observe that we have made several use of the *overline* symbol. We will continue to make occasional use of this symbol (with its various meaning) where they are needed and whenever this will not cause confusion. To summarize: For a reduced commutative unitary ring A, by $\bar{A}$ we mean the algebraic closure (or sometimes called total integral closure) of the ring. For an element $a \in A$ if there is a canonical diagonal monomorphism (as in the above proof) of A into product of rings (or fields) then the canonical image of a is denoted by $\bar{a}$ and in this way we identify A as a subring of this product of rings (this should be clear as long as the domain and codomain are clear to the reader). Similarly, there is a canonical map (as in the above proof) that brings a polynomial $f \in A[x]$ into a polynomial $\bar{f}$ over product of rings (again, we use this only if the domain and codomain of this monomorphism is clear to the reader). With this, we hope that we have avoided unnecessary symbolic clutter in this paper without sacrificing clarity.

Notation. Let A be a subring of a ring B then the group $\mathrm{Aut}(B/A)$ is the set of ring automorphisms of B fixing A, i.e. the group of A-automorphisms of B.

Notation. Let A be a commutative ring and $f \in A[x]$ be a polynomial. Suppose that f is written as follows

$$f(x) := \sum_{i=0}^n a_i x^i$$

then for any ideal $I \trianglelefteq A$, we abuse notation and denote $f \bmod I$ (or $f(x) \bmod I$) as the polynomial

$$f \bmod I := \sum_{i=0}^n (a_i \bmod I) x^i \in (A/I)[x]$$

2 Splitting Rings

Let A be a reduced commutative unitary ring with algebraic closure B and suppose that $f \in A[x]$ is monic. Then for $S := \{b \in B \,:\, f(b) = 0\}$, we want to study the ring extension $A \hookrightarrow A[S]$. We start with a Theorem that claims that if A is Baer than we can at least conclude that B is a finitely generated A-module ...

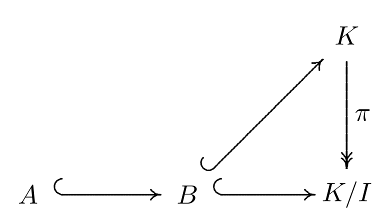
Theorem 2. (Splitting Ring) Let A be a commutative reduced Baer ring. Suppose $B = \bar{A}$ and that $f \in A[x]$ is a non-constant monic polynomial with coefficients in A. Consider the set

$$S := \{b \in B \,:\, f(b) = 0\}$$

then $A[S]$ is a finitely generated module. More specifically, there are $b_1, \ldots, b_n \in B$ (where $n \leq \deg(f)$) such that

$$A[S] = A[b_1, \ldots, b_n]$$

Proof. Define $K := \prod_{\mathfrak{p} \in \mathrm{Min}\, B} \mathrm{Quot}(B/\mathfrak{p})$, we have the following diagram (with all canonical maps)



where I is an ideal maximal (Zorn's Lemma) with the property that $I \cap A = \langle 0 \rangle$. This allows the lower right ring monomorphism in the diagram to be an essential extension. Let $n := \deg(f) \in \mathbb{N}$ and for each $\mathfrak{p} \in \operatorname{Spec} B$ arbitrarily arrange all n (not necessarily distinct) zeros of $f \bmod \mathfrak{p}$ in $\operatorname{Quot}(B/\mathfrak{p})$ (which is a algebraically closed field, see [12] Corollary 1) as $k_{i,\mathfrak{p}}$ for $i = 1, \ldots, n$. Suppose that $\pi_{\mathfrak{p}} : K \twoheadrightarrow \operatorname{Quot}(B/\mathfrak{p})$ is the canonical projection for every $\mathfrak{p} \in \operatorname{Spec} B$. We define $k_i \in K$, for $i = 1, \ldots, n$, by setting the projection of k_i for each $\mathfrak{p}$ to be $\pi_{\mathfrak{p}}(k_i) := k_{i,\mathfrak{p}}$. By the construction we get $f(\pi(k_i)) = 0$ for all $i = 1, \ldots, n$. Thus, since B is totally integrally closed, $\pi(k_i) \in B$ for all $i = 1, \ldots, n$. We define $b_i := \pi(k_i)$ for $i = 1, \ldots, n$ and claim that $A[S] = A[b_1, \ldots, b_n]$ (and because $b_1, \ldots, b_n$ are integral elements of A, $A[S]$ is a finitely generated A-module). Let $b \in S$, then define $e_i \in K$ for $i = 1, \ldots, n$ by

$$\pi_{\mathfrak{p}}(e_i) := \begin{cases} 1 & b \equiv k_{i,\mathfrak{p}} \bmod \mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

Note that the e_i's are well-defined because $b \bmod \mathfrak{p}$ is a zero of $f \bmod \mathfrak{p}$ and the $k_{i,\mathfrak{p}}$'s are all the zeros of $f \bmod \mathfrak{p}$.

Observe now that, in the ring K, $b = \sum_{i=1}^{n} e_i k_i$ and by taking the image of this with respect to π we get $b = \sum_{i=1}^{n} \pi(e_i) b_i$ in the ring K/I. Now e_i is an idempotent in K so the projection $\pi(e_i)$ is an idempotent in K/I. Since B is integrally closed in K/I (this is by the definition of total integral closure, see also [5] Theorem 29.) we also have $b_i, \pi(e_i) \in B$ (all idempotents satisfy the equation $x^2 - x = 0$) for all $i = 1, \ldots, n$. Since A is Baer, it contains all the idempotents of its total integral closure and hence $\pi(e_i) \in A$ (see [17] Lemma 1.6). We thus have shown that $b \in A[b_1, \ldots, b_n]$ for all $b \in S$, hence $A[S] = A[b_1, \ldots, b_n]$. □

We shall call the ring $A[S]$, in the theorem above, the *splitting ring*[1] *of $f \in A[x]$ (over A)*. The above result, however, does not gaurantee us an A-automorphism group that is finitely generated. We give an example of such a group that is not finitely generated and for that we use this very easy Lemma (whose proof we leave to the reader) ...

Lemma 3. Suppose A is a Baer reduced ring, B is a splitting ring of a monic non-constant polynomial $f \in A[x]$ and $b \in B$ is a zero of f then for any $\sigma \in \operatorname{Aut}(B/A)$, $\sigma(b)$ is a zero of f.

Example. In this example, we show a splitting ring B of some $f \in A[x]$ (where A is a Baer reduced ring) that provides a group of A-automorphisms of B, $\operatorname{Aut}(B/A)$, that is not finitely generated. Let $A = \mathbb{Q}^{\mathbb{N}}$ (this is a Baer reduced ring!) and consider $f(x) := x^2 - \bar{2} \in A[x]$. Note that the algebraic closure of A is A-isomorphic to the integral closure of A in $\bar{\mathbb{Q}}^{\mathbb{N}}$ (hints for this can be found in the proof of Theorem 38 [5]). So the ring $B := \mathbb{Q}(\sqrt{2})^{\mathbb{N}}$ is, in fact, the splitting ring of f (observe that $B = A + \sqrt{2}A$). We first claim that the group $G := \operatorname{Aut}(B/A)$ has exponent 2 (thus a torsion group).

Let, for each $j \in \mathbb{N}$, $\pi_j : B \to \mathbb{Q}(\sqrt{2})$ be the canonical projection on the j-th coordinate. Let $\sigma \in G$, then, one checks that, this induces a well-defined field automorphism $\sigma_j \in \operatorname{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ for each $j \in \mathbb{N}$ that maps each $a + b\sqrt{2}$ $(a, b \in \mathbb{Q})$ to $a + b\pi_j(\sigma(\sqrt{2}))$ (by Lemma 3, $\pi_j(\sigma(\sqrt{2})) \in \{\sqrt{2}, -\sqrt{2}\}$). But $\sigma \circ \sigma \in G$ induces (in the same manner) the automorphism $\sigma_j \circ \sigma_j \in \operatorname{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, for all $j \in \mathbb{N}$, and this can only be the identity map. Thus, the group G has a finite exponent 2 and so it is a torsion group (σ was arbitrary and the field automorphism σ_j, for a $j \in \mathbb{N}$, is not necessarily an identity map).

Now suppose that G is finitely generated. Because it is a torsion group with exponent 2, this becomes a trivial Burnside problem (see [21]) and it is known that for this case G must necessarily be finite (and even commutative). This gives us a contradiction because we know that G is an infinite group: Consider, for every $i \in \mathbb{N}$, the element $\sigma \in G$ that induces

$$\sigma_j := \begin{cases} \text{id} & j \neq i \\ \tau & \text{otherwise} \end{cases} \qquad j \in \mathbb{N}$$

[1] this definition is also in accordance to [10], where it probably first appeared

where $\tau \in \text{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\sqrt{2})$ such that $\tau(\sqrt{2}) = -\sqrt{2}$ and $\text{id} : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is the identity map. This gives us countably but infinitely many number of automorphisms in G.

The above example gave us an automorphism group $\text{Aut}(B/A)$ for which the exponent is finite. In general, this is true for any Baer commutative reduced ring A and splitting ring B. To prove this we first show a few nice results about the total quotients $T(A)$ and $T(B)$.

Theorem 4. Let A be a reduced commutative unitary ring. Suppose that A has the property that for any $a \in A$, there exists an idempotent $e \in A$ such that

$$\text{Ann}(aA) = eA$$

then $T(A)$ is von Neumann regular.

Proof. Let $a \in A$ and suppose that $e \in A$ be an idempotent such that $\text{Ann}(aA) = eA$. We claim tha $a + e$ is a regular element of A. Pick a $b \in A$ such that $(a + e)b = 0$. Then after multiplying by a we get $a^2b = 0$. Since A is reduced, we know that $b \in \text{Ann}(aA)$. Thus, there is a $c \in A$ with $ce = b$. This gives us

$$0 = ce(a + e) = ce^2$$

So $ce = 0$ and this leads us to conclude that $ce = b = 0$. So the annihilator of $(a + e)A$ is 0. In other words, $a + e$ is a regular element.

We now know that $a + e$ is invertible in $T(A)$. So we clearly have

$$a + e = (a + e)^2(a + e)^{-1} = (a^2 + e)(a + e)^{-1} \Rightarrow a^2(a + e)^{-1} = a + e - e(a + e)^{-1}$$

But

$$e = e((a + e)(a + e)^{-1}) = (e(a + e))(a + e)^{-1} = e^2(a + e)^{-1} = e(a + e)^{-1}$$

So

$$a^2(a + e)^{-1} = a + e - e = a$$

We can therefore conclude that every element of A has a "quasi-inverse" in $T(A)$.

Now let $\frac{a}{b} \in T(A)$, with $a \in A$ and b a regular element of A. Also set a' to be the quasi-inverse of a in $T(A)$. Then

$$\left(\frac{a}{b}\right)^2 ba' = \frac{1}{b}a^2a' = \frac{1}{b}a = \frac{a}{b}$$

Thus, any element of $T(A)$ has also a quasi-inverse in $T(A)$. In other words, $T(A)$ is a von Neumann regular ring. □

Corollary 5. Let A be a reduced commutative unitary Baer ring then $T(A)$ is Baer, von Neumann regular and $\text{Spec}\, T(A)$ is canonically homeomorphic to the $\text{Min}\, A$.

Proof. The ring A satisfies (because it is Baer) the condition in Theorem 4, so $T(A)$ is von Neumann regular ring. Now, the smallest von Neumann regular intermediate ring of A and $Q(A)$ has a prime spectrum that is canonically isomorphic to $\text{Min}\, A$ (see [16] Theorem 4.4). And the result follows since any two essential extension of a Baer von Neumann regular ring that are von Neumann regular will have homeomorphic prime spectra (see [17] Remark 1.17). □

Remark 6. The Corollary above will give us even more information. If A is a reduced commutative Baer ring and if B is an essential extension of A (then B must necessarily be reduced by Lemma 1.3 in [17]) then, by Storrer's Satz (see [20] 10.1), there is a canonical essential extension $Q(A) \hookrightarrow Q(B)$ and, by [17] Lemma 1.7, $Q(A)$ contains all of the idempotents of $Q(B)$ and thus, by [15] Proposition 2.5 and Storrer's Satz, both A and B are Baer. Thus, by the above Corollary, both $T(B)$ and $T(A)$ are von Neumann regular. Now, there is a canonical essential extension $T(A) \hookrightarrow T(B)$ and if we use [17] Remark 1.17 we conclude that $T(A)$ and $T(B)$ have homeomorphic prime spectra which are homeomorphic to both $\text{Min}\, A$ and $\text{Min}\, B$.

In short: If A is a reduced commutative Baer ring and B is an essential extension of A then

- B is Baer
- $T(A)$ and $T(B)$ are von Neumann regular
- We have the canonical homeomorphisms

$$\operatorname{Min} A \cong \operatorname{Spec} T(A) \cong \operatorname{Min} B \cong \operatorname{Spec} T(B)$$

We first only state the theorem that we want to prove:

Theorem 7. Let A be a Baer ring and B be the splitting ring of a non-constant monic polynomial $f \in A[x]$, then the automorphism group $\operatorname{Aut}(B/A)$ is a torsion group with finite exponent.

Now, before proving the theorem, we give and prove two lemmas on splitting rings that will be used in the main proof of the theorem:

Lemma 8. Let A be a domain and B be the splitting ring of a non-constant monic $f \in A[x]$ then the following holds

- B is a domain
- $\operatorname{Quot}(B)$ is the splitting field of f over $\operatorname{Quot}(A)$

Proof. Any essential extension of A is a domain and a field containing A is an essential extension of A. It easily follows that $\bar{A}$ is the integral closure of A in the algebraic closure of $\operatorname{Quot}(A)$ (see also [12] Corollary 1, p.774). Since the splitting ring lies between $\bar{A}$ and A, it must be a domain.

For the second part, let K be the splitting field of f over $\operatorname{Quot}(A)$ then one easily sees that

$$K = \operatorname{Quot}(A)(b_1, \dots, b_n) = \operatorname{Quot}(A)[b_1, \dots, b_n] = \operatorname{Quot}(A[b_1, \dots, b_n])$$

where b_i, for $i = 1, \dots, n$, are all the zeros of f in the algebraic closure of $\operatorname{Quot}(A)$. This proves the Lemma since $B = A[b_1, \dots, b_n]$. □

Lemma 9. Suppose that B is the splitting ring of a non-constant monic $f \in A[x]$ over a Baer ring A. It follows that for any $\mathfrak{p} \in \operatorname{Min} B$, $B/\mathfrak{p}$ is the splitting ring of $A/(\mathfrak{p} \cap A)$.

Proof. In this proof, for simplicity, (because A is Baer) we identify all the minimal prime spectra of rings between A and $\bar{A}$ with $\operatorname{Spec} T(A)$. Let $\mathfrak{p} \in \operatorname{Min} B$, then we have $B/\mathfrak{p} = A[S]/\mathfrak{p} = (A/\mathfrak{p})[S/\mathfrak{p}]$ where

$$S := \{b \in B \,:\, f(b) = 0\}$$

Clearly any element in $S/\mathfrak{p}$ is a zero of $f \bmod \mathfrak{p}$, so it suffices to show that $S/\mathfrak{p}$ contains all the zeros of $f \bmod \mathfrak{p}$. Let $k \in \bar{A}/\mathfrak{p}$ be a zero of $f \bmod \mathfrak{p}$ (observe that $\bar{A}/\mathfrak{p}$ is integrally closed and has an algebraically closed quotient field, see e.g. [12] Corollary 1). There is a $b \in \bar{A}$ such that b is canonically mapped to k. Define the disjoint clopen sets (recall that $T(A)$ is von Neumann regular)

$$\begin{aligned} U_1 &:= \{\mathfrak{q} \in \operatorname{Spec} T(A) \,:\, f(b) \bmod \mathfrak{q} \equiv 0\} \\ U_2 &:= \operatorname{Spec} T(A) \backslash U_1 \end{aligned}$$

that cover $\operatorname{Spec} T(A)$ and from this we can define idempotents in e_1 and e_2 in $T(A)$ by

$$e_i \bmod \mathfrak{q} := \begin{cases} 1 & \mathfrak{q} \in U_i \\ 0 & \mathfrak{q} \notin U_i \end{cases} \qquad i = 1, 2$$

This implies that $c := be_1 + se_2 \in S$ for any $s \in S$ (since $f(c) \equiv 0 \bmod \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Spec} T(A)$). Furthermore, $c \bmod \mathfrak{p} = k$ and we so are done. □

Finally the proof of Theorem 7…

Proof of Theorem 7. Denote $G := \mathrm{Aut}(B/A)$ and suppose $\sigma \in G$. Since σ is an automorphism, for any minimal prime ideal $\mathfrak{p} \in \mathrm{Min}\, B$ the set $\sigma(\mathfrak{p})$ is also a minimal prime ideal of B. Because A is Baer and B is an essential extension of A, we know that there is a canonical homeomorphism from $\mathrm{Min}\, B$ to $\mathrm{Min}\, A$ (see Remark 6) given by

$$\mathrm{Min}\, B \to \mathrm{Min}\, A \qquad \mathfrak{q} \mapsto \mathfrak{q} \cap A$$

Since $\mathfrak{p} \cap A$ is a minimal prime ideal of A and σ is an A-automorphism we have

$$\sigma(\mathfrak{p} \cap A) = \mathfrak{p} \cap A \subset \sigma(\mathfrak{p}) \cap A \subset \sigma(\mathfrak{p})$$

which would mean, by the above homeomorphism, that $\sigma(\mathfrak{p}) = \mathfrak{p}$.

Set $\mathfrak{p}_A := \mathfrak{p} \cap A$ then, since $\sigma(\mathfrak{p}) = \mathfrak{p}$, σ induces a well-defined $A/\mathfrak{p}_A$-automorphism

$$B/\mathfrak{p} \to B/\mathfrak{p} \qquad b \bmod \mathfrak{p} \mapsto \sigma(b) \bmod \mathfrak{p}$$

This in turn canonically induces a $\mathrm{Quot}(A/\mathfrak{p}_A)$-automorphism

$$\mathrm{Quot}(B/\mathfrak{p}) \xrightarrow{\sim} \mathrm{Quot}(B/\mathfrak{p})$$

The minimal prime spectra $\mathrm{Min}\, \bar{A}, \mathrm{Min}\, B$ and $\mathrm{Min}\, A$ are all homeomorphic (see Remark 6) and they can all be canonically identified. Now suppose $\mathfrak{p} \in \mathrm{Min}\, \bar{A}$ and, for simplicity, write $\mathfrak{p}_A := \mathfrak{p} \cap A$ and $\mathfrak{p}_B := \mathfrak{p} \cap B$. By the Lemmas 8 and 9, $B/\mathfrak{p}_B \cong A[b_1, \ldots, b_n]/\mathfrak{p}$ for some $b_i \in B$ such that $b_i \bmod \mathfrak{p}$ (i=1,…,n) are all the roots of $f \bmod \mathfrak{p}$ in the algebraic closed field $\mathrm{Quot}(\bar{A}/\mathfrak{p})$ and $\mathrm{Quot}(B/\mathfrak{p}_B)$ is the splitting field of $f \bmod \mathfrak{p}$ over $\mathrm{Quot}(A/\mathfrak{p}_A)$.

Now, let us write for short $K_\mathfrak{p} := \mathrm{Quot}(A/\mathfrak{p}_A)$ and $L_\mathfrak{p} := \mathrm{Quot}(B/\mathfrak{p}_B)$. We have already seen that $\sigma \in \mathrm{Aut}(A/B)$ induces a $K_\mathfrak{p}$-automorphism $\sigma_\mathfrak{p} \in \mathrm{Aut}(L_\mathfrak{p}/K_\mathfrak{p})$. Similarly, for any $k \in \mathbb{N}$, $\sigma^k \in \mathrm{Aut}(A/B)$ will induce $\sigma_\mathfrak{p}^k \in \mathrm{Aut}(L_\mathfrak{p}/K_\mathfrak{p})$. But $L_\mathfrak{p}$ is the splitting field for $f \bmod \mathfrak{p}$, so if n is the degree of f, the order of $\sigma_\mathfrak{p}$ will divide $n!$ (e.g. [8] Theorem 3 p.176 discusses this). Thus $\sigma^{n!}$ induces the identity $L_\mathfrak{p} \to L_\mathfrak{p}$. This is true for any $\mathfrak{p} \in \mathrm{Min}\, \bar{A}$.

In other words, for any $\mathfrak{p} \in \mathrm{Min}\, \bar{A}$ (or $\mathfrak{p}_A \in \mathrm{Min}\, A$) and for any $\sigma \in \mathrm{Aut}(B/A)$ we have the equation

$$\sigma^{n!}(b) \equiv b \bmod \mathfrak{p} \qquad \forall b \in B$$

Since B is reduced, $\sigma^{n!}(b) = b$ for all $b \in B$ and this means that $n!$ is an exponent of the group $\mathrm{Aut}(A/B)$.

□

3 Generalized Artin Schreier Theorem

We recall and reformulate Remark 109 in [5]

Remark 10. Let A be a reduced ring and B be an overring of A, then there is a canonical map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$. This map has the property that its image contains $\mathrm{Min}\, A$. For the proof: Let $\mathfrak{p} \in \mathrm{Min}\, A$ and consider the multiplicative set $S = A \backslash \mathfrak{p}$, then from the monomorphism $A \hookrightarrow B$ we get the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & S^{-1}B \end{array}$$

$S^{-1}A$ is a field (because $\mathfrak{p}$ is minimal) and its only prime ideal, which is $\{0\}$, corresponds to $\mathfrak{p}$ in $\mathrm{Min}\, A$. Hence, the canonical image of

$$\mathrm{Spec}\, S^{-1}B \cong \bigcap_{s \in S} D_B(s) \subset \mathrm{Spec}\, B$$

in $\operatorname{Spec} A$ is $\{\mathfrak{p}\}$. We see that an arbitrary $\mathfrak{p} \in \operatorname{Min} A$ belongs in the image of the canonical map $\operatorname{Spec} B \to \operatorname{Spec} A$.

Recall also Theorem 101 in [5]

Theorem 11. Let A be a Baer reduced ring, then A is normal iff for any $\mathfrak{p} \in \operatorname{Spec} T(A)$ we have $A/(\mathfrak{p} \cap A)$ is integrally closed in $T(A)/\mathfrak{p}$.

Proof. See [5] Theorem 101. □

We can combine a few results in [5] to arrive to the following Theorem

Theorem 12. Let A be a Baer reduced ring and suppose that $T(A)$ is a subring of a von Neumann regular real closed B. If, furthermore, A is integrally closed in B then $T(A)$ is a real closed von Neumann regular ring.

Proof. This follows immediately from Theorem 99(iii) and Theorem 34 in [5]. □

Corollary 13. A real Baer ring B is real closed iff for every minimal prime ideal $\mathfrak{p} \in \operatorname{Min} B$ one has $B/\mathfrak{p}$ is a real closed integral domain.

Proof. "$\Rightarrow$" If B is real closed then it is integrally closed in $Q(B)$ and furthermore $Q(B)$ is real closed (see [5] Remark 28). Now, by Theorem 12, $T(B)$ is also a real closed von Neumann regular ring. It follows from [19] Proposition 2 and Theorem 11 that $B/\mathfrak{p}$ is a real closed integral domain for any minimal prime ideal $\mathfrak{p}$ in $\operatorname{Spec} B$ ($\mathfrak{p} \in \operatorname{Min} B$ is a restriction of a prime ideal in $T(B)$, see Remark 6).

"$\Leftarrow$" Since B is Baer, $T(B)$ is von Neumann regular. So, by the hypothesis, the residue domains of B with respect to prime ideals in $\operatorname{Min} B$ is integrally closed in their real closed quotient fields (see [19] Proposition 2), and these quotient fields are the residue fields of $T(B)$. Thus the residue fields of the Baer von Neumann regular ring $T(B)$ are real closed and so (see [5] Theorem 34) $T(B)$ is real closed. Finally, Theorem 11 implies that B is normal and so all conditions for [22] Theorem 3 are satisfied (B is real closed iff $Q(B)$ is real closed and B is integrally closed in $Q(B)$. Note that $Q(B)$ is also the complete ring of quotient of $T(B)$.) and so B is real closed. □

The minimal prime ideals of real rings are actually quite important in real algebra. Here is one Proposition that gives us an insight on this

Proposition 14. Let A be a real ring and $\mathfrak{p} \in \operatorname{Min} A$ then $A/\mathfrak{p}$ is a real ring (i.e. the prime ideal $\mathfrak{p}$ is a *real ideal*).

Proof. Consider the canonical homorphism $A \to A_{\mathfrak{p}}$. If $a \notin \mathfrak{p}$ then $\frac{a}{1}$ is invertible and if $a \in \mathfrak{p}$ then by [13] Corollary 2.2 there exists an $b \notin \mathfrak{p}$ such that $ab = 0$ and therefore $\frac{a}{1} \equiv 0$ in $A_{\mathfrak{p}}$. Thus, the kernel of this morphism is $\mathfrak{p}$ and therefore by [4] Proposition 2.1.1 $\mathfrak{p}$ is a convex prime ideal (convexity considered as a subset of the poset A with partial order induced by sum of squares of A, but this is also valid with respect to any partial ordering of A). So by [2] Proposition 4.3.8, $\mathfrak{p}$ is also a real prime ideal. □

In the following easy lemma we see the relationship between integral extension of (containing) domains and their quotient fields and how essential extension plays a role in this relationship...

Lemma 15. Let A and B be integral domains and $A \subset B$ as rings, then

i. If B is integral over A, then $\operatorname{Quot}(B)$ is algebraic over $\operatorname{Quot}(A)$.

ii. If B is an essential extension of A and A is integrally closed in B, then $\operatorname{Quot}(A)$ is algebraically closed in $\operatorname{Quot}(B)$.

Proof. i. The set of elements in $\mathrm{Quot}(B)$ that is algebraic over $\mathrm{Quot}(A)$ form a field, define this as K. Then $B \subset K$ since B is integral over A. Moreover $B \subset K \subset \mathrm{Quot}(B)$ implies that $K = \mathrm{Quot}(B)$. Thus $\mathrm{Quot}(B)$ is algebraic over $\mathrm{Quot}(A)$.

ii. Let $\frac{b}{c}$ be an element of $\mathrm{Quot}(B)$, with $b, c \in B$ and $c \neq 0$. Because B is essential over A, without loss of generality we may assume that $c \in A$. Suppose furthermore that $\frac{b}{c}$ is an algebraic element of $\mathrm{Quot}(A)$. Then there is a polynomial

$$f(T) = T^n + \sum_{i=0}^{n-1} T^i \frac{a_i}{x} \in \mathrm{Quot}(A)[T]$$

with $a_i \in A$ and $x \in A \backslash \{0\}$ and such that $\frac{b}{c}$ a zero of f. So

$$f(b/c) = \left(\frac{b}{c}\right)^n + \sum_{i=0}^{n-1} \frac{b^i a_i}{x c^i} = 0$$

Now multiply $f(b/c)$ by $(cx)^n$ (which is not 0, since c and x are non-zero), then

$$c^n x^n f(b/c) = (bx)^n + \sum_{i=0}^{n-1} (bx)^i c^{n-i} x^{n-i-1} a_i = 0$$

and therefore $bx \in B$ is a zero of

$$T^n + \sum_{i=0}^{n-1} b_i T^i \in A[T]$$

where $b_i := c^{n-i} x^{n-i-1} a_i \in A$ for $i = 0, \ldots, n-1$. But A is integrally closed in B, so $bx \in A$. This implies that

$$b/c = bx/cx \in \mathrm{Quot}(A)$$

□

Theorem 16. Let A be a normal Baer real ring. If $\bar{A}$ is a finitely generated A-module, then

i) A is a real closed ring

ii) If A is von Neumann regular, then $A[\sqrt{-1}] = \bar{A}$

Proof. i) Let $\mathfrak{p}$ be a minimal prime ideal of A, then both $A/\mathfrak{p}$ and $\mathrm{Quot}(A/\mathfrak{p})$ are real (see Proposition 14). There is a $\tilde{\mathfrak{p}} \in \mathrm{Spec}\,\bar{A}$ such that $\tilde{\mathfrak{p}} \cap A = \mathfrak{p}$ (see Remark 10). Since $\bar{A}/\tilde{\mathfrak{p}}$ is a finite integral extension of $A/\mathfrak{p}$, we see that $\mathrm{Quot}(\bar{A}/\tilde{\mathfrak{p}})$ is a finite field extension of $\mathrm{Quot}(A/\mathfrak{p})$ (see Lemma 15 or [1] Proposition 2.1.10). We know by [12] Theorem 1 that $\mathrm{Quot}(\bar{A}/\tilde{\mathfrak{p}})$ is an algebraically closed field and so by the classical Artin-Schreier Theorem (see [14] §1 Theorem in p.18) $\mathrm{Quot}(A/\mathfrak{p})$ is a real closed field.

By Theorem 11 and Remark 6, for any $\mathfrak{p} \in \mathrm{Min}\,A$, $A/\mathfrak{p}$ is integrally closed in its quotient field. This quotient field, we have shown, is real closed. Thus, for any $\mathfrak{p} \in \mathrm{Min}\,A$ the integral domain $A/\mathfrak{p}$ is real closed ([19] Proposition 2). Employ Corollary 13 to conclude that the ring A is also real closed.

ii) Let i (or $\sqrt{-1}$) be a zero of $T^2 + 1 \in A[T]$ in the splitting ring of A. Note that $A[i]$ and $\bar{A}$ are von Neumann regular because an integral extension of a von Neumann regular ring that is reduced is also von Neumann regular (see [17] Lemma 1.9). A being Baer implies that $\mathrm{Spec}\,A$,$\mathrm{Spec}\,A[i]$ and $\mathrm{Spec}\,\bar{A}$ are canonically homeomorphic. We have also previously seen that, for all $\mathfrak{p} \in \mathrm{Min}\,A =$

Spec A and (unique) $\tilde{p} \in \operatorname{Min} A[i] = \operatorname{Spec} \bar{A}[i]$ lying over $\mathfrak{p}$, $A/\mathfrak{p} = \operatorname{Quot}(A/\mathfrak{p})$ is real closed and have algebraic closure

$$(A/\mathfrak{p})[i \bmod \tilde{p}] = (A/\mathfrak{p})[\sqrt{-1}] = A[i]/\tilde{\mathfrak{p}}$$

Thus, for all $\tilde{\mathfrak{p}} \in \operatorname{Spec} A[i]$, $A[i]/\tilde{p}$ is an algebraically closed field and this is a characterization of algebraically closed von Neumann regular reduced rings (see [12] Proposition 5) and thus $A[i] \cong \tilde{A}$ □

Unfortunately, we do not yet know whether the second part of the theorem above is true for reduced rings in general. The complication lies on the fact that we used the characterisation for algebraically closed rings (i.e. for each prime ideal, residue domains are algebraically closed). We do not know this in general, because normality does not in general hold for non-minimal prime ideals (Proposition 5 in [12] requires algebraic closedness for *all* residue domain). For von Neumann regular rings, we had the convenience that the residue domains where themselves fields.

In the following, we show why we cannot remove the precondition *Baer* from the theorem above...

Example. Define the $X := \beta\mathbb{N} \times \{0,1\}$ with $\{0,1\}$ having the discrete topology. Define also $Y := X/\sim$ where

$$(x,1) \sim (y,0) \Leftrightarrow x, y \in \beta\mathbb{N}\backslash\mathbb{N} \text{ and } x = y$$

with the usual quotient topology. Then clearly both X and Y are Stone spaces with X being extremally disconnected. For brevity, we write the image of any $(x,i) \in X$ in Y also as (x,i). And we define $\psi : X \to Y$ to be the canonical surjection from X to Y.

Now we state a few facts:

- Y is not extremally disconnected because the closure of the open set

  $$\{(x,0) \,:\, x \in \mathbb{N}\} \subset Y$$

  is not open in Y.

- Let K be a real closed field and consider the von Neumann regular rings

  $$A_Y := \{f : Y \to K \,:\, f^{-1}(k) \text{ is open for all } k \in K\}$$

  and

  $$A_X := \{f : X \to K \,:\, f^{-1}(k) \text{ is open for all } k \in K\}$$

  Then both A_Y and A_X are von Neumann regular rings (see [12] p.779) with prime spectra Y and X respectively. Because of the surjection ψ, we know that the (canonical) map defined by

  $$\phi : A_Y \to A_X \qquad f \mapsto f \circ \psi$$

  is injective. It is clear that ϕ is a ring monomorphism.

- We claim that if $f : X \to K$ is in A_X then $g : X \to K$ defined by

  $$g(x_1, x_2) := \begin{cases} f(x_1, x_2) & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases}$$

  is in A_Y.

- Define now $e : X \to K$ the following way

  $$e(x_1, x_2) := \begin{cases} 1 & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases}$$

  We claim that $e \in A_X$. Let $k \in K$, then if $k \notin \{0,1\}$ we have $f^{-1}(k) = \emptyset$ which is clearly open in X. For $k \in \{0,1\}$ we have

  $$f^{-1}(0) = \{(x_1, x_2) \,:\, x_1 \in \beta\mathbb{N},\, x_2 = 1\}$$

$$f^{-1}(1) = \{(x_1, x_2) \,:\, x_1 \in \beta\mathbb{N},\, x_2 = 0\}$$

which are also open in X. Thus $e \in A_X$.

- We claim that $A_Y[e] = A_X$
  Let $f \in A_X$, then define $g_1 : X \to K$ by

  $$g_1(x_1, x_2) := f(x_1, 0)$$

  and $g_2 : X \to K$ by

  $$g_2(x_1, x_2) := f(x_1, 1)$$

  First we claim that g_1 and g_2 are in A_Y, but this is clear since for any

  $$g_i(x, 0) = g_i(x, 1) \qquad \forall x \in \beta\mathbb{N}, i = 1, 2$$

  We can then easily check that $f = g_1 e + g_2(1 - e)$ and thus conclude that $f \in A_Y[e]$.

Because X is extremally disconnected and Y is not, we know that A_X is Baer and A_Y is not Baer. It is also clear that e is a rational element of A_Y. Now define $e' : X \to K$ the following way

$$e'(x_1, x_2) := \begin{cases} 1 & x_1 \in \mathbb{N} \text{ and } x_2 = 0 \\ 0 & \text{else} \end{cases}$$

Then $e' \in A_X$ because $e'(x, 0) = e'(x, 1)$ for all $x \in \beta\mathbb{N}\backslash\mathbb{N}$ and for any $k \in K$ the set $e'^{-1}(k)$ is open. Finally we also see that e is a rational element of A_Y, since $e' \cdot e \in A_Y\backslash\{0\}$. Since we are dealing with von Neumann regular rings A_X and A_Y we know that these rings are normal. Because of the preceding Theorem we also know that $A_X[\sqrt{-1}]$ is a total integral closure of A_Y (since A_X is Baer). We also note that e is not in $A_Y[\sqrt{-1}]$ and so the result of the Theorem above does not hold if we remove the condition that the ring should be Baer.

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