MOBILE ICOSAPODS

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Abstract. Pods are mechanical devices constituted of two rigid bodies, the base and the platform, connected by a number of other rigid bodies, called legs, that are anchored via spherical joints. It is possible to prove that the maximal number of legs of a mobile pod, when finite, is 20. In 1904, Borel designed a technique to construct examples of such 20-pods, but could not constrain the legs to have base and platform points with real coordinates. We show that Borel's construction yields all mobile 20-pods, and that it is possible to construct examples where all coordinates are real.

Introduction

A multipod is a mechanical linkage consisting of two rigid bodies, called the *base* and the *platform*, and a number of rigid bodies, called *legs*, connecting them. Each leg is attached to base and platform with spherical joints (see Figure 1), so platform points are constrained to lie on spheres — the center of each sphere is then the base point connected to the respective leg. If the platform can move respecting the constraints imposed by the legs we say that the multipod is *mobile*.

Note that multipods are also studied within *Rigidity Theory* as so-called *body-bar frameworks* [1], as two rigid bodies (platform and base) are connected by multiple bars (legs). Mobile multipods correspond to flexible body-bar frameworks, whose study is of great practical interest e.g. for protein folding [2].

We can model the possible configurations of a multipod using direct isometries of $\mathbb{R}^3$, by associating to every configuration the isometry, which maps a reference configuration into the respective one. At this point one can consider a *motion*, namely a one-dimensional set of direct isometries, and try to construct a multipod moving according to this motion. This approach has a long history; there are motions allowing multipods with infinitely many legs, but among those that allow only a finite number of legs, the maximal number is 20. This was proved by Schoenflies, see Remark 2.6 below. Borel proposed a construction for icosapods (namely multipods with 20 legs) leading to linkages whose motion is line-symmetric, i.e. whose elements are involutions, namely rotations by 180° around a line (see Figure 1).

This paper provides two results on icosapods. First, we show that all mobile icosapods (with very mild restrictions) are instances of Borel's construction. Second, we exhibit a mobile icosapod that is an instance of Borel's construction (Borel, in fact, obtained equations for the base and platform points of a mobile icosapod, but could not prove the existence of solutions in $\mathbb{R}^3$). The second part is closely related to the theory of quartic spectrahedra, which has been studied in [3].

Section 1 provides an historical overview of line-symmetric motions. In Section 2 we set up the formalism and the objects that are needed for our approach to the problem; in particular we recall a compactification of the group of direct isometries that has already been used by the authors to deal with problems on multipods, and we show how the constraints imposed by legs can be interpreted as a duality between the space of legs and the space of direct isometries. In Section 3 we use these tools

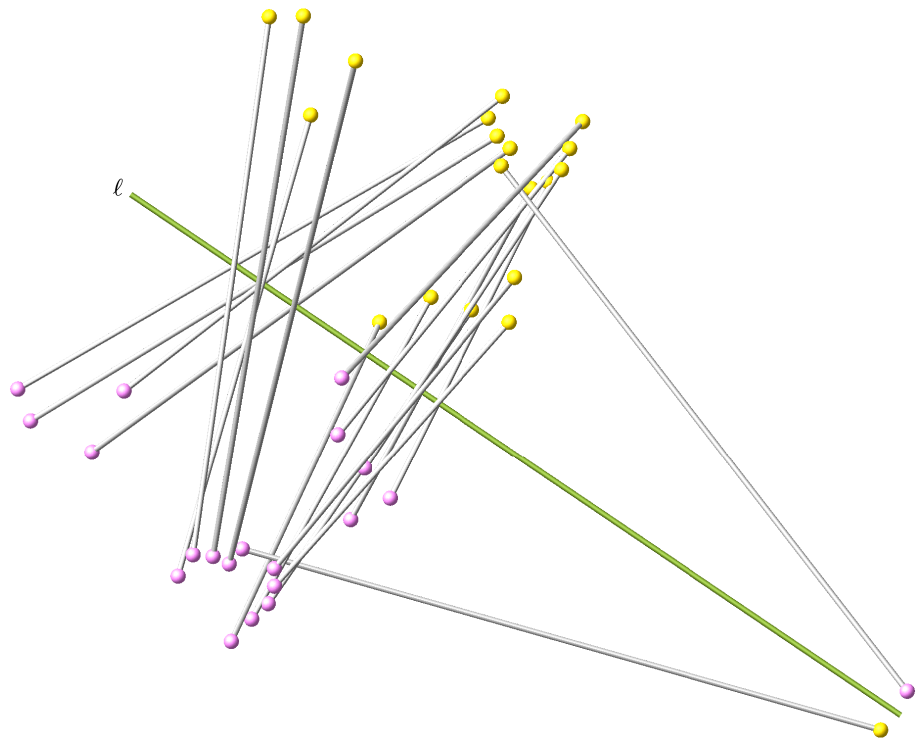

FIGURE 1. Base points (pink) and platform points (yellow) of an icosapod. Notice that for every leg there is a symmetric one obtained by rotating the first one by 180° around the axis ℓ. The symmetry reverses the role of base and platform points.

to prove that, under certain generality conditions, mobile icosapods admit line-symmetric motions. In Section 4 we show how it is possible to construct example of mobile icosapods employing results in the theory of quartic spectrahedra.

1. Review on line-symmetric motions

Krames [4] studied special one-parametric motions, obtained by reflecting the moving system ς in the generators of a ruled surface in the fixed system Σ. This ruled surface is called the *base surface* for the motion. He showed some remarkable properties of these motions (see [4]), which led him to name them *Symmetrische Schrotung* (in German). This name was translated to English as *symmetric motion* by Tölke [5], *Krames motion* or *line-symmetric motion* by Bottema and Roth [6, page 319]. As each *Symmetrische Schrotung* has the additional property that it is equal to its inverse motion (cf. Krames [7, page 415]), it could also be called *involutory motion*. In this paper we use the name *line-symmetry* as it is probably the most commonly used term in today's kinematic community.

Further characterizations of line-symmetric motions (beside the cited one of Krames [4]) where given by Tölke [5], Bottema and Roth [6, Chapter 9, § 7], Selig and Husty [8] and Hamann [9]. If one uses the so-called Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ to describe isometries, then it is possible to characterize line-symmetric motions algebraically in the following way. Given such a motion, there always exist a Cartesian system of coordinates, or *frame* $(o; x, y, z)$ for the moving system ς and a Cartesian frame $(O; X, Y, Z)$ for the fixed system Σ so that $e_0 = f_0 = 0$ holds for the elements of the motion. With this choice of coordinates, the latter are rotations by 180° around lines; the Study coordinates $(e_1 : e_2 : e_3 : f_1 : f_2 : f_3)$ of these isometries coincide with the Plücker coordinates of the lines.

1.1. **Historical results on line-symmetric motions with spherical paths.** In this paper we study line-symmetric motions that are solutions to the still open problem posed by the French Academy of Science for the *Prix Vaillant* of the year 1904 (cf. [10]): *"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."* Borel and Bricard were awarded the prize for their papers [11] and [12] containing partial solutions, and therefore this is also known as the *Borel Bricard (BB) problem.*

1.1.1. *Krames's results.* Krames [7, 13, 14] studied some special motions already known to Borel and Bricard in more detail and stated the following theorem [7, Satz 6]:

Theorem 1.1. *For each line-symmetric motion, that contains discrete,* 1 *or* 2*-dimensional spherical paths, the set* $\mathfrak{f}$ *of points with spherical trajectories is congruent (direct isometry) to the set* $\mathfrak{F}$ *of corresponding sphere centers.*

Moreover Krames noted in [7, page 409] that[1] *"... most of the solutions given by Borel and Bricard are line-symmetric motions. In each of these motions both geometers detected this circumstance by other means, without using the above mentioned result" (Theorem 1.1).* In the following we will take a closer look at the papers [11, 12], which shows that the latter statement is not entirely correct.

1.1.2. *Bricard's results.* Bricard studied these motions in [12, Chapitre VIII]. His first result in this context (see end of [12, § 32, page 70]) reads as follows (adapted to our notation):

Dans toutes les solutions auxquelles on sera conduit, les figures liées $\mathfrak{F}$ *et* $\mathfrak{f}$ *seront évidemment égales et semblablement placées par rapport aux deux trièdres* $(O; X, Y, Z)$ *et* $(o; x, y, z)$*.*

In the remainder of [12, Chapitre VIII] he discussed some special cases, which also yield remarkable results, but he did not give further information on the general case.

1.1.3. *Borel's results.* Borel discussed in [11, Case Fb] exactly the case $e_0 = f_0 = 0$ and he proved in [11, Case Fb1] that in general a set of 20 points are located on spherical paths but without giving any result on the reality of the 20 points. Moreover he studied two special cases in Fb2 and Fb3.

Borel did not mention the geometric meaning of the assumption $e_0 = f_0 = 0$. He only stated at the beginning of case F [11, page 95] that the moving frame $(o; x, y, z)$ is parallel to the frame obtained by a reflection of the fixed frame $(O; X, Y, Z)$ in a line. This corresponds to the weaker assumption $e_0 = 0$. He added that this implies the same consequences as already mentioned in [11, page 47, case C], which reads as follows (adapted to our notation):

... dans le cas où les trièdres sont symétriques par rapport à une droite, si deux courbes sont représentées par des équations identiques, l'une en X, Y, Z*, l'autre en* x, y, z*, elles sont symétriques par rapport à cette droite.*

But Borel did not mention, neither in case Fb1 nor in his conclusion section, that $\mathfrak{f}$ with $\#\mathfrak{f} = 20$ is congruent to $\mathfrak{F}$ (contrary to other special cases e.g. Fb3, where the congruence property is mentioned explicitly.)

[1] The following extract as well as Theorem 1.1 has been translated from the original German by the authors.

1.2. **Review of line-symmetric self-motions of hexapods.** We denote the platform points of the i-th leg in the moving system ς by p_i and its corresponding base points in the fixed system Σ by P_i.

For a generic choice of the geometry of the platform and the base as well as the leg lengths d_i the hexapod can have up to 40 configurations. Under certain conditions it can also happen that the direct kinematic problem has no discrete solution set but an n-dimensional one with $n > 0$. Clearly these so-called self-motions of hexapods are solutions to the BB problem.

In practice hexapods appear in the form of Stewart-Gough manipulators, which are 6 degrees of freedom parallel robots. In these machines the leg lengths can be actively changed by prismatic joints and all spherical joints are passive.

Moreover a hexapod (resp. Stewart-Gough manipulator) is called *planar* if the points $p_1, \ldots, p_6$ are coplanar and also the points $P_1, \ldots, P_6$ are coplanar; otherwise it is called *non-planar*. In the following we review those papers where line-symmetric self-motions of hexapods are reported.

1.2.1. *Non-planar hexapods with line-symmetric self-motions.* Line-symmetric motions with spherical paths already known to Borel [11] and Bricard [12] (and also discussed by Krames in [7, 14]) were used by Husty and Zsombor-Murray [15] and Hartmann [16] to construct examples of (planar and non-planar) hexapods with line-symmetric self-motions.

Point-symmetric hexapods with congruent platform and base possessing line-symmetric self-motions were given in [17, Theorem 11]. Further non-planar hexapods with line-symmetric self-motions can be constructed from overconstrained pentapods with a linear platform [18].

1.2.2. *Planar hexapods with line-symmetric self-motions.* All self-motions of the original Stewart-Gough manipulator were classified by Karger and Husty [19]. Amongst others they reported a self-motion with the property $e_0 = 0$ (see [19, page 208, last paragraph]), *"which has the property that all points of a cubic curve lying in the plane ... and six additional points out of this plane have spherical trajectories. This seems to be a new case of a BB motion, not known so far."* Based on this result, Karger [20, 21] presented a procedure for computing further *"new self-motions of parallel manipulators"* of the type $e_0 = 0$, where the points of a planar cubic $\mathfrak{c}$ have spherical paths.

Another approach was taken by Nawratil in his series of papers [22, 23, 24, 25], by determining the necessary and sufficient geometric conditions for the existence of a 2-dimensional motion such that three points in the xy-plane of ς move on three planes orthogonal to the XY-plane of Σ (3-fold Darboux condition) and two planes orthogonal to the xy-plane of ς slide through two fixed points located in the XY-plane of Σ (2-fold Mannheim condition). It turned out that all these so-called type II Darboux-Mannheim motions are line-symmetric. Moreover a geometric construction of a 12-parametric set of planar Stewart-Gough platforms (cf. [25, Corollary 5.4]) with line-symmetric self-motions was given. It was also shown that the algorithm proposed by Karger in [20, 21] yields these solutions.

While studying the classic papers of Borel and Bricard for this historical review we noticed that the solution set of the BB problem mentioned in the last two paragraphs was already known to these two French geometers; cf. [11, Case Fb3] and [12, Chapter V] (already reported by Bricard in [26, page 21]). But in contrast to the above listed approaches (of Karger and Nawratil) both of them assumed that the motion with spherical trajectories is line-symmetric. Each of them additionally discovered one more property:

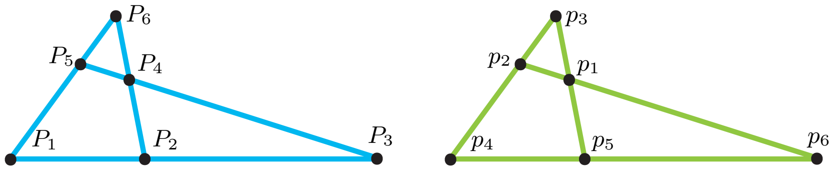

FIGURE 2. Illustration of Duporcq's complete quadrilaterals.

- Borel pointed out that there exist further 8 points (all 8 can be real) with spherical trajectories. This set of points splits in four pairs, which are symmetric with respect to the carrier plane of the cubic $\mathfrak{c}$.
- Bricard showed the following: If we identify the congruent planar cubics of the platform and the base, i.e. $\mathfrak{c} = \mathfrak{C}$, then the tangents in a corresponding point pair P and p with respect to $\mathfrak{c} = \mathfrak{C}$ intersect each other in a point of the cubic $\mathfrak{c} = \mathfrak{C}$ (P and p form a so-called Steinerian couple).

Bricard communicated his result (published in [26, page 21]) to Duporcq, who gave an alternative reasoning in [27], which sank into oblivion over the past 100 years. Only a footnote in the conclusion section of Borel's work [11] points to Duporcq's proof (but not to the original work of Bricard [26]), which is based on the following remarkable motion (see Figure 2):

Let $P_1, \ldots, P_6$ and $p_1, \ldots, p_6$ be the vertices of two complete quadrilaterals, which are congruent. Moreover the vertices are labelled in a way that p_i is the opposite vertex of P_i for $i \in \{1, \ldots, 6\}$. Then there exist a two-parametric line-symmetric motion where each p_i moves on a sphere centered at P_i.

It can be easily checked that this configuration of base and platform points corresponds to an architecturally singular hexapod (e.g. [28] or [29]). As architecturally singular manipulators are redundant we can remove any leg — w.l.o.g. we suppose that this is the sixth leg — without changing the direct kinematics of the mechanism. Therefore the resulting pentapod $P_1, \ldots, P_5$ and $p_1, \ldots, p_5$ also has a two-parameter, line-symmetric, self-motion.

Note that this pentapod yields a counter-example to Theorem 4.2 of [30] and as a consequence also the work [31] is incomplete as it is based on this theorem. For the erratum to [30] please see [32] and the addendum to [31] is given in [33].

Remark 1.2. If we assume for this pentapod that the line $[P_1P_2]$ is the ideal line of the fixed plane (so $[p_4p_5]$ is the ideal line of the moving plane) then we get exactly the conditions found in [22], which the points and ideal points of the plane's normals must fulfill in order to get a type II Darboux-Mannheim motion. Note that P_2, P_3 can also be complex conjugates (so p_5, p_6 are complex conjugate too).

1.2.3. *Computer search for mobile hexapods.* In [34], Geiss and Schreyer describe a rather non-standard way to find mobile hexapods. They set up an algebraic system of equations equivalent to mobility, and then try random candidates, with coordinates in a finite field of small size, by computer. After collecting statistical data indicating the existence of a family of real mobile hexapods, they try a (computationally more expensive) lifting process in the most promising cases. The method is extremely powerful and could still be used for finding new families of mobile hexapods. However, the family reported in [34] can be seen as an instance of Borel's family Fb1. The line symmetry is not apparent because the method starts by guessing 6 legs, which may not form a line symmetric configuration; only if one adds the remaining 14 legs, or at least all real legs among them, one obtains a symmetric configuration. The question posed in Problem 5 in [34] is easily answered from this viewpoint: two legs have the same lengths because they are conjugated by line symmetry.

2. Isometries, legs and bond theory

This section illustrates the concepts and techniques that will be needed to carry on our analysis. From now on, by an *n-pod* we mean a triple $(\vec{p}, \vec{P}, \vec{d})$ where $\vec{p}, \vec{P}$ are n-tuples of vectors in $\mathbb{R}^3$ (respectively, platform and base points), and $\vec{d}$ is an n-tuple of positive real numbers. Multipods with 20 legs are called *icosapods*.

Remark 2.1. The description of admissible configurations that we adopt allows independent choices for two coordinate systems, one for the base and one for the platform: this is why also the leg lengths have to be included in the definition of a pod. Moreover, we do not require that the identity belongs to the admissible configurations.

2.1. **Isometries and leg space.** Throughout this paper we use a compactification of the group SE_3 of direct isometries that was introduced in [35] and in [36]. We briefly recall this construction.

Consider a point a lying on a sphere centered at b and with radius d, the equation expressing this coincidence can be written,

$$\|a - b\|^2 - d^2 = 0.$$

Expanding this gives

$$\big(\langle a, a\rangle + \langle b, b\rangle - d^2\big) - 2\,\langle a, b\rangle \;=\; 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. This can be written as a scalar product of two 5-vectors, one containing information on the sphere the other only information about the point,

$$\big(-2b^t,\, (\langle b, b\rangle - d^2),\, 1\big) \begin{pmatrix} a \\ 1 \\ \langle a, a\rangle \end{pmatrix} = 0.$$

Now a direct isometry σ, of $\mathbb{R}^3$ can be described by a pair (M, y), where M is an orthogonal matrix with $\det(M) = 1$ and y is the image of the origin under the isometry. If the point a remains on the sphere after the action of σ, then we have, $\|\sigma(a) - b\|^2 - d^2 = 0$. In terms of the 5-vectors we can write this condition as,

$$\big(-2b,\, (\langle b, b\rangle - d^2),\, 1\big) \begin{pmatrix} M & y & 0 \\ 0 & 1 & 0 \\ 2x^t & r & 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \\ \langle a, a\rangle \end{pmatrix} = 0,$$

where the matrix represents the action of the isometry and we have defined $x := -M^t y = -M^{-1} y$ and $r := \langle x, x\rangle = \langle y, y\rangle$. Notice that this determines a 5-dimensional representation of the group SE_3, displaying the direct isometries as a subgroup of the group of conformal transformations of space.

If $a, b \in \mathbb{R}^3$ and $d \geq 0$, then the condition $\|\sigma(a) - b\|^2 - d^2 = 0$ on a direct isometry σ is linear, and in particular has the following form:

$$\big(\langle a, a\rangle + \langle b, b\rangle - d^2\big)\, h + r - 2\,\langle a, x\rangle - 2\,\langle y, b\rangle - 2\,\langle Ma, b\rangle \;=\; 0, \tag{1}$$

where the homogenizing variable h has been included. We will usually refer to Equation (1) as the *sphere condition* imposed by (a, b, d).

So, if we take coordinates $m_{11}, \ldots, m_{33}$, x_1, x_2, x_3, y_1, y_2, y_3 and r, h in $\mathbb{P}^{16}_{\mathbb{C}}$, a direct isometry defines a point in projective space satisfying $h \neq 0$ and

$$
\begin{aligned}
MM^t \;=\; M^t M \;=\; h^2 \cdot \mathrm{id}_{\mathbb{R}^3}, \quad \det(M) \;=\; h^3, \\
M^t y + hx \;=\; 0, \quad Mx + hy \;=\; 0, \\
\langle x, x \rangle \;=\; \langle y, y \rangle \;=\; r\, h.
\end{aligned}
\tag{2}
$$

The equations in (2) define a variety X in $\mathbb{P}^{16}_{\mathbb{C}}$, whose real points satisfying $h \neq 0$ are in one to one correspondence with the elements of SE_3. A direct calculation shows that X is a variety of dimension 6 and degree 40.

Given an n-pod $\Pi = (\vec{p}, \vec{P}, \vec{d})$, we can consider its *configuration space*, namely the set of direct isometries σ in SE_3 such that $\|\sigma(p_i) - P_i\|^2 - d_i^2 \;=\; 0$ for all $i \in \{1, \ldots, n\}$. The fact that Equation (1) is linear in the coordinates of $\mathbb{P}^{16}_{\mathbb{C}}$ means that the configuration space of Π can be compactified as the intersection of X with a linear space Λ, determined by the n conditions imposed by its legs; we denote such variety by K_Π. We say that the pod Π is *mobile* if the intersection $K_\Pi(\mathbb{R}) \cap \{h \neq 0\}$ — where we denoted by $K_\Pi(\mathbb{R})$ the real points of K_Π — has (real) dimension greater than or equal to one. Notice that if Π is mobile, then K_Π has (complex) dimension greater than or equal to one.

Notation. From now on, by the expression *mobility one icosapod* we mean a 20-pod with mobility one which cannot be obtained by removing legs from a multipod with the same configuration set.

Remark 2.2. For reasons of completeness it should be noted that system in Equation (2) was already used by Mourrain [37, page 293] to prove that a non-mobile pod has at most 40 configurations. Similar systems of equations were also used by Lazard [38, page 179] and Wampler [39, Equation (2)] for the same task.

For our purposes, it is useful to introduce another 16-dimensional projective space, playing the role of a dual space of the one containing X, where the duality is given by a bilinear version of Equation (1). We start by introducing a new quantity, called *corrected leg length*, defined as $l := \langle a, a \rangle + \langle b, b \rangle - d^2$, so that Equation (1) becomes

$$
l\, h + r - 2\, \langle a, x \rangle - 2\, \langle y, b \rangle - 2\, \langle Ma, b \rangle \;=\; 0.
$$

We think of the points $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ as points in $\mathbb{P}^3_{\mathbb{C}}$ by introducing two extra homogenization coordinates a_0 and b_0. In this way, the pair (a, b) can be considered as a point in the Segre variety $\Sigma_{3,3} \cong \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$; the latter is embedded in $\mathbb{P}^{15}_{\mathbb{C}}$, where we take coordinates $\{z_{ij}\}$ so that the points of $\Sigma_{3,3}$ satisfy $z_{ij} = a_i\, b_j$ for some $(a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}^3_{\mathbb{C}}$. If we homogenize Equation (1) with respect to the coordinates $\{z_{ij}\}$ and l, then we get

$$
\begin{aligned}
& l\, h + z_{00} r - 2\, (z_{10} x_1 + z_{20} x_2 + z_{30} x_3) - \\
& \qquad\qquad - 2\, (z_{01} y_1 + z_{02} y_2 + z_{03} y_3) - 2 \sum_{i,j=1}^{3} m_{ij}\, z_{ij} \;=\; 0
\end{aligned}
\tag{3}
$$

Notice that the left hand side of Equation (3) is a bilinear expression in the coordinates (h, M, x, y, r) and in the coordinates (z, l). We denote this expression by BSC (for *bilinear sphere condition*). Hence, if we denote by $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ the projective space with coordinates (z, l), then we obtain a duality between $\mathbb{P}^{16}_{\mathbb{C}}$ and $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ sending a point $(h_0, M_0, x_0, y_0, r_0) \in \mathbb{P}^{16}_{\mathbb{C}}$ to the hyperplane in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ of equation $\mathrm{BSC}(h_0, M_0, x_0, y_0, r_0, z, l) = 0$, and a point $(z_0, l_0) \in \check{\mathbb{P}}^{16}_{\mathbb{C}}$ to the hyperplane in $\mathbb{P}^{16}_{\mathbb{C}}$ of equation $\mathrm{BSC}(h, M, x, y, r, z_0, l_0) = 0$.

Remark 2.3. Suppose that $(z_0, l_0) \in \check{\mathbb{P}}^{16}_{\mathbb{C}}$ belongs to the cone Y with vertex $(0 : \cdots : 0 : 1)$ over the Segre variety $\Sigma_{3,3}$ — namely $(z_0)_{ij} = a_i\, b_j$ for some $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$. Suppose furthermore that $a_0 = b_0 = 1$ and $a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l_0 \geq 0$. Then from the definition of BSC we see that the hyperplane $\mathrm{BSC}(h, M, x, y, r, z_0, l_0) = 0$ in $\mathbb{P}^{16}_{\mathbb{C}}$ is the same hyperplane defined by Equation (1) where we take $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $d = \sqrt{a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l}$.

Remark 2.3 indicates that the cone Y in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ plays a sort of dual role to the one of the compactification X in $\mathbb{P}^{16}_{\mathbb{C}}$, and we will exploit this in our arguments. Since the Segre variety $\Sigma_{3,3}$ has dimension 6 and degree 20, we see that Y has dimension 7 and degree 20.

Definition 2.4. Let $C \subseteq X$ be a curve. We define the *leg set* L_C as the set of all points $(z, l) \in Y$ such that the BSC — instantiated at (z, l) — holds for all points in C. The leg set is a compactification of the set of all triples $(a, b, d) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ such that the image of a under any point in C lying in the image of SE_3 (hence considered as an isometry) has distance d from b.

Proposition 2.5. *Let $C \subseteq X$ be a curve. If L_C has only finitely many complex points, then its cardinality is at most* 20*. If L_C has exactly* 20 *points, then the linear span of L_C in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ is a projective subspace of dimension* 9*.*

Proof. By construction L_C is defined by linear equations as a subset of Y; in other words $L_C = Y \cap \mathrm{span}(L_C)$. Then the statement follows from general properties of linear sections of projective varieties. In fact, any linear subspace of codimension less than 7 intersects Y in a subvariety of positive dimension, hence $\dim\big(\mathrm{span}(L_C)\big) \leq 9$. A general linear subspace of dimension 9 intersects Y in $\deg(Y) = 20$ points. In order to prove that $\dim\big(\mathrm{span}(L_C)\big) = 9$ when L_C has 20 points we assume the contrary, that $\dim\big(\mathrm{span}(L_C)\big) < 9$. Then we take a general linear superspace Λ of $\mathrm{span}(L_C)$ of dimension 9. It intersects Y again in 20 points, which must coincide with L_C. On the other hand, if the Hilbert series of Y is $\frac{P(t)}{(1-t)^{16}}$, then the Hilbert series of $\Lambda \cap Y$ is $\frac{P(t)}{(1-t)^{9}}$, but this contradicts the fact that $\Lambda \cap Y = L_C$ is contained in a linear space of dimension strictly smaller than 9. □

In order to show the maximal number of 20 intersections can be achieved in Proposition 2.5, we need to find a curve $C \subseteq X$ such that the bilinear sphere conditions of its points define a linear subspace in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ of dimension 9. This is equivalent to asking $\dim\big(\mathrm{span}(C)\big) = 15 - 9 = 6$. We will deal with this problem in Section 3.

Remark 2.6. The proof of Proposition 2.5 also gives a simple alternative proof for the number of solutions (over $\mathbb{C}$) of the spatial Burmester problem, which reads as follows: Given seven poses $\varsigma_1, \ldots, \varsigma_7$ of a moving system ς, determine all points of ς, that are located on a sphere in all seven poses.

The given poses correspond to seven points in X, which span in the general case a $\mathbb{P}^6_{\mathbb{C}}$. Therefore its dual space is of dimension 9, and so intersects Y in exactly 20 points.

This number was first computed by Schoenflies in [40, page 148] and confirmed by Primrose (see [41, footnote 3]) as well as by Wampler *et al.* [42, Section 5]. We conclude noticing that the first solution of the spatial Burmester problem using an approach based on polynomial systems was presented by Innocenti [43], who also gave an example with 20 real solutions.

2.2. **Bond theory.** The second ingredient for our arguments is *bond theory*, a technique for analyzing mobile multipods that was introduced and developed (in the form we need here) in [36]. If K_Π is the configuration curve of a multipod Π of mobility one, then the *bonds* of Π are defined as the intersections of K_Π with the hyperplane $H = \{h = 0\}$. Note that such intersections always arise in conjugate complex pairs since K_Π is a real algebraic variety and has no real points.

Recall, that the elements of SE_3 are in 1-to-1 correspondence with the real points of the variety X so long as $h \neq 0$. So the bonds are points in the closure of SE_3 in this model of the group. The set $B = H \cap X$ has dimension 5, and its points, although they do not represent isometries, still have geometric meaning as conditions imposed on the legs: we mean that if the configuration set K_Π passes through a point in B than base and platform points of Π must satisfy a certain condition. The variety B can be partition into five subsets, which differ by the condition imposed on base and platform points of multipods. Detail of these subsets can be found in [36], here we simply list the possibilities. We just point out that we are not going to use the properties of inversion and similarity points, we report them here only for the sake of completeness.

vertex: the only real point in B; it is never contained in a configuration curve.

inversion points: if a multipod Π has a configuration set passing through an inversion point, then there exists two directions L and R in $\mathbb{R}^3$ such that if we project the base of Π orthogonally along L and the platform of Π orthogonally along R, we obtain two tuples in the plane that correspond via an inversion (depending on the boundary point).

similarity points: if a multipod Π has a configuration set passing through a similarity point, then there exists two directions L and R in $\mathbb{R}^3$ such that if we project the base of Π orthogonally along L and the platform of Π orthogonally along R, we obtain two tuples in the plane that correspond via a similarity (depending on the boundary point).

butterfly points: these correspond to a pair of lines in $\mathbb{R}^3$, one for the base and one for the platform; a multipod Π whose configuration set passes through a butterfly point either has the base point on the base line or the platform point on the platform line.

collinearity points: these correspond to lines; if a multipod Π has a configuration set passing through a collinearity points, then either its platform points or its base points are collinear.

A mobile icosapod cannot have butterfly bonds (namely bonds that are butterfly points), nor can it have collinearity bonds as this would imply that there are at least 10 legs with collinear base points or platform points. Then all points of the line carrying those base or platform points would be base or platform points for a multipod with the same configuration set. We will not consider these cases in our discussion.

Inversion points are smooth points of X, and their tangent spaces are contained in H. Consequently every motion passing an inversion point is tangent to the hyperplane H at this point. In particular, if a multipod Π of mobility one has only inversion bonds, the degree of its configuration curve C is twice the number of inversion bonds: the degree can be computed by intersecting C with H, and there we see only pairs of double intersections.

Consider the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ keeping only the coordinates h and m_{ij} for $i, j \in \{1, 2, 3\}$. The real points of the open subset defined by $h \neq 0$ in the image of X is the group variety SO_3, and the image itself is a subvariety X_m of degree 8 that is isomorphic to the Veronese embedding of $\mathbb{P}^3_{\mathbb{C}}$ by quadrics. This follows from the bijection between points $(e_0 : e_1 : e_2 : e_3) \in \mathbb{P}^3_{\mathbb{R}}$ and orthogonal matrices (see

[44, Section 4.5]) given by

$$
\begin{array}{c}
(e_0 : e_1 : e_2 : e_3) \\
\updownarrow \\
\frac{1}{e_0^2+e_1^2+e_2^2+e_3^2}\begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2e_1e_2-2e_0e_3 & 2e_0e_2+2e_1e_3 \\ 2e_1e_2+2e_0e_3 & e_0^2-e_1^2+e_2^2-e_3^2 & -2e_0e_1+2e_2e_3 \\ -2e_0e_2+2e_1e_3 & 2e_0e_1+2e_2e_3 & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}.
\end{array}
\tag{4}
$$

In fact, if we consider the Veronese variety that is the image of the morphism

$$
\begin{array}{ccc}
\mathbb{P}^3_{\mathbb{C}} & \longrightarrow & \mathbb{P}^9_{\mathbb{C}} \\
(e_0 : e_1 : e_2 : e_3) & \mapsto & (e_0^2 : e_1^2 : e_2^2 : e_3^2 : e_0\, e_1 : e_0\, e_2 : e_0\, e_3 : e_1\, e_2 : e_1\, e_3 : e_2\, e_3)
\end{array}
$$

and we apply the projective automorphism given by the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\tag{5}
$$

then a direct computation shows that we obtain X_m. The coordinates of $\mathbb{P}^3_{\mathbb{C}}$ are called *Euler parameters* and denoted by e_0, e_1, e_2, e_3. The center of the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^9_{\mathbb{C}}$ intersects X in the union of the sets of similarity points, collinearity points and the vertex.

2.3. **The subvariety of involutions.** We focus our attention on a particular subvariety of X, the compactification of the set of involutions in SE_3. Involutions in SE_3 are rotations of 180° around a fixed axis, so their compactification — which we will denote by X_{inv} — is a 4-dimensional subvariety of X, because the family of lines in $\mathbb{R}^3$ is 4-dimensional. One reason why involutions are particularly useful in the creation of mobile pods is that if p and P are a platform and a base point of a pod Π, and $\sigma \in K_{\Pi}$ is an involution in the configuration space of Π, then this means that $\|\sigma(p) - P\| = d$, where d is the distance between p and P; on the other hand, since σ is an involution we have $\|\sigma(P) - p\| = d$. This means that if all isometries in the configuration space of Π are involutions, then we can swap the roles of base and platform points and obtain "for free" new legs not imposing any further restriction to the possible configurations of Π.

If $\sigma = (M, y)$ is an involution, then $M = M^t$, that is to say M is symmetric, and $y = x$. Hence we can consider the subvariety

$$
\bigl\{ (h : M : x : y : r) \in X \ : \ M = M^t \text{ and } x = y \bigr\}.
$$

One verifies that this subvariety has two irreducible components, namely the isolated point corresponding to the identity and another one of dimension 4, which is cut out by a further linear equation, namely $m_{11} + m_{22} + m_{33} + h = 0$.

Definition 2.7. The subvariety of X defined by the equations $M = M^t$ and $x = y$ and $m_{11} + m_{22} + m_{33} + h = 0$ is denoted X_{inv}.

3. Mobile icosapods are line-symmetric

In this section we will show that if an icosapod of mobility one admits an irreducible configuration curve, then its motion is line-symmetric (Theorem 3.10). We start by translating this concept into our formalism. Recall from [36, Section 2.2] that the group SE_3 acts on its compactification X: every isometry in SE_3 determines a projective automorphism of $\mathbb{P}^{16}_{\mathbb{C}}$ leaving X invariant.

Definition 3.1. Let $C \subseteq X$ be a curve. Then C is called an *involutory motion* if $C \subseteq X_{\text{inv}}$. The curve C is called a *line-symmetric motion* if there exists an isometry τ such that the automorphism associated to τ maps C inside X_{inv}.

From Proposition 2.5 we know that the configuration curve of an icosapod spans a linear subspace of dimension 6. To get an overview of possible examples of irreducible curves $C \subseteq X$ with $\dim\bigl(\operatorname{span}(C)\bigr) = 6$, we consider the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ described at the end of Subsection 2.2. Let $C_m \subseteq X_m$ be the projection of C, which can be either a point or a curve. It is possible to prove (see [45]) that if C_m is a point, then there exists a multipod with infinitely many legs admitting C as configuration set. Since we are interested in pods with finitely many legs, from now we suppose that C_m is a curve. Let $C_e \subseteq \mathbb{P}^3_{\mathbb{C}}$ be its isomorphic preimage under the Veronese map.

Proposition 3.2. *If $C \subseteq X$ is an irreducible curve such that $\dim\bigl(\operatorname{span}(C)\bigr) = 6$, then C_e is either planar or a twisted cubic.*

Proof. The projection of $\operatorname{span}(C)$ in $\mathbb{P}^9_{\mathbb{C}}$ is a linear subspace of dimension at most 6. Hence the ideal of C_m contains at least 3 linear independent linear forms. Hence the ideal of C_e contains at least 3 linear independent quadratic forms. If any of such quadratic form is reducible, namely splits into the union of two planes, the statement follows; thus from now on we can suppose that all of them are irreducible. Then the intersection of the zero sets of two of these quadrics is a quartic curve D such that $C_e \subseteq D$. It cannot happen that $C_e = D$, because this would contradict the fact that there are three independent quadrics passing through C_e. Therefore the degree of C_e can only be 1, 2 or 3. If the degree is 1 or 2, then C_e is planar; if the degree is 3, then C_e is either planar or a twisted cubic, because by construction C_e is irreducible. The proposition then follows. $\square$

From now on, since we aim for a result on mobile icosapods, we will consider curves C allowing exactly 20 legs, that is satisfying the following condition:

$$(\dagger) \qquad \begin{cases} C \text{ is an irreducible real curve with real points,} \\ L_C \text{ consists of exactly 20 real finite points.} \end{cases}$$

Here by real “finite” points we mean that their z_{00}-coordinates are not zero; in other terms, such points determine pairs of base and platform points in $\mathbb{R}^3$ (and not at infinity).

Remark 3.3. Notice that condition (†) does not comprise every mechanical device that one could name a “mobile icosapod”: there may exist a device admitting infinitely many complex points in its leg space, of which only 20 are real and finite; we believe that such situation cannot occur, but we were not able to provide an argument for this. Still, we believe that condition (†) is a good compromise because it will allow a uniform treatment of the topic.

Remark 3.4. Notice that condition (†) implies that $\dim\bigl(\operatorname{span}(C)\bigr) = 6$. Moreover, from Section 2.2 it follows that C does not pass through any butterfly or collinearity point.

Lemma 3.5. *Suppose that $C \subseteq X$ satisfies condition (†). Then C_e cannot be a cubic (neither a twisted cubic, nor a plane cubic).*

Proof. Suppose that C satisfies condition (†) and C_e is a twisted cubic. Then the curve C_m, isomorphic to C_e under the Veronese embedding, is a rational normal sextic, and therefore spans a linear space of dimension 6. This means that the projection $C \longrightarrow C_m$ is a projective isomorphism — since by Remark 3.4 also the curve C spans a linear space of dimension 6. This forces the center of the projection to be disjoint from span(C). By Section 2.2, this implies that C does not admit similarity or collinearity points. Recall from Remark 3.4 that the curve C does not admit butterfly points, so the only ones left are the inversion points. However, since the degree of C is twice the number of inversion points, we would get 3 of them, and this is not possible, since they occur in conjugate pairs.

Now, suppose that C_e is a planar cubic. Then the curve C_m spans a linear space of dimension 5. This means that the center of the projection $C \longrightarrow C_m$ intersects span(C) in a single point. Since similarity points occur in pairs, as remarked at the beginning of Section 2.2, such intersection cannot be a similarity point; as before, Remark 3.4 ensures that the curve C does not admit butterfly or collinearity points, so C has only inversion points. Hence we can conclude as in the previous case. □

We focus therefore on curves C satisfying (†) such that C_e is planar of degree different from 3. First we rule out the case when C_e is a line or a conic.

Lemma 3.6. *Suppose that $C \subseteq X$ satisfies condition (†). Then C_e cannot be a line.*

Proof. If C_e is a line the corresponding motion can only be a Schoenflies motion[2]. These motions with points moving on spheres where previously studied by Husty and Karger in [46]. It is not difficult to see that no discrete solution can exist, as any leg can be translated along the axis of the Schoenflies motion without restricting the self-motion. Therefore we always end up with an ∞-pod. □

Lemma 3.7. *Suppose that $C \subseteq X$ satisfies condition (†). Then C_e cannot be a conic.*

Proof. Assume that C_e is a conic. The bilinear spherical condition BSC determines a subspace $\check{\mathbb{P}}^9_{\mathbb{C}} \subseteq \check{\mathbb{P}}^{16}_{\mathbb{C}}$ as dual to the linear projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^9_{\mathbb{C}}$. By a direct inspection of Equation (3) one notices that the subvariety $Y_\infty \subseteq Y$ composed of those pairs (a, b) of points with $a_0 = b_0 = 0$ (namely legs for which both the base and the platform point are at infinity) is contained in $\check{\mathbb{P}}^9_{\mathbb{C}}$. The dimension of Y_∞ is 5 and its degree is 6, since it is a cone over the Segre variety $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$.

Consider now the set of linear forms in $\mathbb{P}^9_{\mathbb{C}}$ vanishing on C_m: this is a vector space of dimension 5, since it is isomorphic to the vector space of all quadratic forms on $\mathbb{P}^3_{\mathbb{C}}$ that vanish along C_e. In this way we get a linear space $\check{\mathbb{P}}^4_{\mathbb{C}} \subseteq \check{\mathbb{P}}^9_{\mathbb{C}}$. The intersection $\check{\mathbb{P}}^4_{\mathbb{C}} \cap Y_\infty \subseteq L_C$ is non-empty and is in general constituted of 6 points. Thus the number of real legs not at infinity is at most 14, and this contradicts the assumption that L_C has 20 real finite points. □

For the remaining cases, when C_e is a planar curve of degree greater than or equal to 4, we want to prove that there exists an isometry τ such that the image of C under the corresponding projective automorphism is contained in X_{inv}. Recall from Section 2.2 that we denoted $e_0, \ldots, e_3$ the coordinates of the $\mathbb{P}^3_{\mathbb{C}}$ where C_e lives. We may assume without loss of generality that $e_0 = 0$ holds for the points of C_e; this can be achieved by a suitable rotation of the coordinate frame of the platform — specifically by acting on C with a suitable rotation — since by assumption C_e is planar. In terms of the coordinates of X, this means that we can apply an automorphism of $\mathbb{P}^{16}_{\mathbb{C}}$ induced by an isometry so that the points of C satisfy $m_{ij} = m_{ji}$ and $m_{11} + m_{22} + m_{33} + h = 0$: this follows from the relations between the variables $(h : M)$ and the variables $(e_0 : e_1 : e_2 : e_3)$ (see in particular Equations (4) and (5) when $e_0 = 0$).

[2]These motions can be composed by a rotation about a fixed axis and an arbitrary translation.

We can use the assumption $e_0 = 0$ in order to simplify the embedding. Let X_1 be the intersection of X with the linear space

$$\big\{(h : M : x : y : r) \in \mathbb{P}^{16}_{\mathbb{C}} \,:\, m_{ij} = m_{ji} \text{ and } m_{11} + m_{22} + m_{33} + h = 0\big\}.$$

Notice that X_1 is the set-theoretical preimage of the locus $\{e_0 = 0\}$ under the projection $X \dashrightarrow \mathbb{P}^3_{\mathbb{C}}$, where such projection is obtained composing the projection to $\mathbb{P}^9_{\mathbb{C}}$ on the $(h : M)$-coordinates with the inverse of the Veronese embedding $\mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^9_{\mathbb{C}}$. We project X_1 from the point $(0 : \cdots 0 : 1)$ in $\mathbb{P}^{16}_{\mathbb{C}}$ — the only singular point of X of order 20 — obtaining a subvariety X_2 of $\mathbb{P}^{15}_{\mathbb{C}}$, which is actually contained in a linear subspace isomorphic to $\mathbb{P}^{11}_{\mathbb{C}}$ because of the linear equations imposed on X_1.

If we express the coordinates h and m_{ij} for $i, j \in \{1, 2, 3\}$ in terms of the Euler coordinates e_1, e_2, e_3, and apply the coordinate change

$$p_i \;=\; x_i + y_i, \quad q_i \;=\; x_i - y_i, \qquad \text{for } i \in \{1, 2, 3\},$$

then we obtain a map from X_2 to the weighted projective space $\mathbb{P}_{\mathbb{C}}(\vec{1}, \vec{2})$ (see [47, Example 10.27] for a reference for weighted projective space). Here $\vec{1} = (1, 1, 1)$ and $\vec{2} = (2, 2, 2, 2, 2, 2)$ and we take coordinates e_1, e_2, e_3 of weight 1 and $p_1, p_2, p_3, q_1, q_2, q_3$ of weight 2. The image of X_2 is a weighted projective variety $Z \subseteq \mathbb{P}_{\mathbb{C}}(\vec{1}, \vec{2})$ of dimension 5 defined by the equations

$$e_1\, p_1 + e_2\, p_2 + e_3\, p_3 \;=\; p_1\, q_1 + p_2\, q_2 + p_3\, q_3 \;=\; e_1\, q_2 - e_2\, q_1 \;=\; e_1\, q_3 - e_3\, q_1 \;=\; e_2\, q_3 - e_3\, q_2 \;=\; 0,$$

as a direct computation using computer algebra confirms. Therefore, for a curve $C \subseteq \mathbb{P}^{16}_{\mathbb{C}}$ for which C_e is planar and satisfies $e_0 = 0$ we get a map $C \longrightarrow C_z \subseteq Z$ that is the composition of a projection, a linear change of variables and a Veronese map.

Remark 3.8. At the beginning of the section we pointed out that isometries determine automorphisms of $\mathbb{P}^{16}_{\mathbb{C}}$ leaving X invariant. Notice that the automorphisms corresponding to translations leave X_1 invariant, since its equations comprise only the rotational part of isometries. Therefore translations also act on Z.

Lemma 3.9. *Suppose that $C \subseteq X$ satisfies condition (†). Let $C_z \subseteq Z$ be the image of the curve C under the previously defined maps. Then there exists a translation of the platform such that the corresponding automorphism maps C_z to a curve C'_z whose points satisfy $q_1 = q_2 = q_3 = 0$.*

Proof. From the discussion so far we may infer that $\deg(C_e) > 3$. We are especially interested in the q-vector, so let $W \subseteq \mathbb{P}_{\mathbb{C}}(1, 1, 1, 2, 2, 2)$ be the projection of Z to the e and q-coordinates and let C_w be the image of C_z under such projection. The set W has dimension 4, and its equations are

$$e_1\, q_2 - e_2\, q_1 \;=\; e_1\, q_3 - e_3\, q_1 \;=\; e_2\, q_3 - e_3\, q_2 \;=\; 0. \tag{6}$$

By a direct inspection of the map $C \longrightarrow C_w$ one notices that forms of weighted degree 2 on C_w correspond to linear form on C. It follows that the vector space of weighted degree 2 forms on C_w has dimension at most 7. There are 9 forms of weighted degree 2 on $\mathbb{P}_{\mathbb{C}}(1, 1, 1, 2, 2, 2)$ and they are all linear independent as forms on W because the latter is defined by equations of weighted degree 3. Hence C_w satisfies at least 2 equations $E_1 = E_2 = 0$ of weighted degree 2.

By construction, the polynomials E_i are of the form $E_i = L_i\,(\vec{q}) + Q_i\,(\vec{e})$, where L_i is linear and Q_i is quadratic. Notice that $L_1\,(\vec{q})\,L_2\,(\vec{e}) - L_1\,(\vec{e})\,L_2\,(\vec{q})$ vanishes on W, because it is a multiple of the polynomials in Equation (6). Therefore on C_w we have

$$E_1\,(\vec{e}, \vec{q})\; L_2\,(\vec{e}) - E_2\,(\vec{e}, \vec{q})\; L_1\,(\vec{e}) \;=\; Q_1\,(\vec{e})\; L_2\,(\vec{e}) - Q_2\,(\vec{e})\; L_1\,(\vec{e}) \;=\; 0.$$

The latter is a cubic equation only in the variables $\vec{e}$, thus it is satisfied by C_e. On the other hand, C_e is a planar curve of degree greater than 3, so C_e cannot satisfy a nontrivial cubic equation. Therefore we conclude that $Q_1(\vec{e})\, L_2(\vec{e}) - Q_2(\vec{e})\, L_1(\vec{e})$ is zero on $\mathbb{P}^2_{\mathbb{C}}$. Since L_1 and L_2 cannot be proportional (otherwise we would be able to obtain from E_1 and E_2 a quadratic equation in $\vec{e}$ satisfied by C_e) we conclude by unique factorization that L_i is a factor of Q_i for $i \in \{1, 2\}$. Hence $Q_i(\vec{e}) = L(\vec{e})\, L_i(\vec{e})$ for some linear polynomial L.

From Equation (6) we infer that $L_1(\vec{q})\, e_j = L_1(\vec{e})\, q_j$ for $j \in \{1, 2, 3\}$. Since E_1 is zero on C_w, we have $-L_1(\vec{q}) = L(\vec{e})\, L_1(\vec{e})$ on C_w. Multiplying by e_j the last equation yields:

$$-L_1(\vec{e})\, q_j \;=\; -L_1(\vec{q})\, e_j \;=\; L(\vec{e})\, L_1(\vec{e})\, e_j,$$

implying that $q_j = L(\vec{e})\, e_j$ holds on C_w for $j \in \{1, 2, 3\}$.

One can verify that the automorphism corresponding to the translation by a vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ acts on the coordinates of $\mathbb{P}_{\mathbb{C}}(1, 1, 1, 2, 2, 2)$ by sending

$$(\vec{e}, \vec{q}) \;\mapsto\; (\vec{e}, \vec{q} + \ell_{\vec{a}}\, \vec{e}),$$

where $\ell_{\vec{a}} = a_1 e_1 + a_2 e_2 + a_3 e_3$. Hence, if $L(\vec{e}) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, it is enough to apply to C_w the automorphism corresponding to the translation by the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ to get that $q_1 = q_2 = q_3 = 0$ holds on C_w. This proves the statement. $\square$

We are now ready to prove our main result.

Theorem 3.10. *Let Π be a mobile icosapod such that its configuration curve K_Π is irreducible and satisfies* (†). *Then K_Π is a line-symmetric motion.*

Proof. From the discussion so far (Proposition 3.2, Lemma 3.5 and the paragraph following Lemma 3.7) we know that it is possible to apply a projective automorphism to K_Π so that the equations $m_{ij} = m_{ji}$ and $m_{11} + m_{22} + m_{33} + h = 0$ hold. Hence we only need to ensure $x = y$. However, in the new embedding in $\mathbb{P}_{\mathbb{C}}(1, 1, 1, 2, 2, 2)$ defined in Lemma 3.9 those equations correspond to $q_1 = q_2 = q_3 = 0$, so Lemma 3.9 shows the claim. $\square$

Remark 3.11. From Theorem 3.10 the set of base and platform points of the icosapod possesses a line-symmetry during the complete self-motion. But this property holds for any pod with a line-symmetric self-motion (see Theorem 1.1). As a consequence one could call these mechanical linkages "line-symmetric icosapods" by analogy to the "line-symmetric Bricard octahedra".

4. Construction of real icosapods

Borel proposed the construction of line-symmetric icosapods simply by intersecting X_{inv} with a general linear subspace T of dimension 7 in $\mathbb{P}^{10}_{\mathbb{C}}$, as explained in Section 4.1. Since X_{inv} is a variety of codimension 6 and degree 12 in $\mathbb{P}^{10}_{\mathbb{C}}$, the intersection $C = X_{\mathrm{inv}} \cap T$ is an irreducible curve of degree 12. Indeed, C is a canonical curve of genus 7, as one can read off from the Hilbert series of X_{inv}. The projection C_e of C to the Euler parameters is a planar sextic. Recall that the leg set L_C is the intersection of Y with the dual of $\mathrm{span}(C)$, a linear subspace of codimension 7; this is, in general, a set of 20 complex points. It was not previously known whether there are examples with 20 real legs, and the goal of this section is to show that there are instances of such curves C for which this is the case. We reduce the problem to a question on spectrahedra whose answer is well-known.

4.1. **Borel's construction.** We rephrase Borel's construction using the terminology and concepts introduced in this paper. Let S be the linear subspace defined by the equations $M = M^t$ and $x = y$. Notice that S has projective dimension 10 and contains X_{inv}, although $\mathrm{span}(X_{\mathrm{inv}})$ is a linear subspace of dimension 9. The restriction of the bilinear sphere condition from Equation (3) can be written in a symmetric way as

$$l\,h + z_{00}r - 2\sum_{i=1}^{3} s_{0i}x_i - 2\sum_{i=1}^{3} z_{ii}m_{ii} - 2\sum_{1\le i<j}^{3} s_{ij}m_{ij} \;=\; 0, \tag{7}$$

where $s_{ij} = z_{ij} + z_{ji}$ for $1 \le i < j \le 3$. We denote this equation by SBSC, for *symmetric bilinear sphere condition.* It defines a duality between S and a linear subspace $\check{\mathbb{P}}^{10}_{\mathbb{C}} \subseteq \check{\mathbb{P}}^{16}_{\mathbb{C}}$ whose projective coordinates are l,$z_{00}, \ldots, z_{33}$,$s_{01}, \ldots, s_{23}$. The intersection of the leg variety Y with such a $\check{\mathbb{P}}^{10}_{\mathbb{C}}$ parametrizes pairs of legs obtained by swapping the roles of the base and the platform points. Denote by $\pi\colon \check{\mathbb{P}}^{10}_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^{9}_{\mathbb{C}}$ the projection defined by removing the l-coordinate. We denote by Y_{inv} the image of the map $\alpha\colon \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}} \longrightarrow \check{\mathbb{P}}^9_{\mathbb{C}}$,

$$\big((a_0 : \cdots : a_3), (b_0 : \cdots : b_3)\big) \mapsto (\underbrace{a_0\, b_0}_{z_{00}} : \cdots : \underbrace{a_3\, b_3}_{z_{33}} : \underbrace{a_0\, b_1 + a_1\, b_0}_{s_{01}} : \cdots : \underbrace{a_2\, b_3 + a_3\, b_2}_{s_{23}}).$$

One can easily prove that Y_{inv} is nothing but the projection under π of the intersection $Y \cap \check{\mathbb{P}}^{10}_{\mathbb{C}}$. Note that α is a $2:1$ map, since $\alpha(a,b) = \alpha(b,a)$ for all $a, b \in \mathbb{P}^3_{\mathbb{C}}$. Because of this, it might happen that two pairs of complex points are sent by α to a real point of Y_{inv}.

For any curve $C \subseteq X_{\mathrm{inv}}$, the leg set L_C is equal to the intersection of the linear space $\widetilde{\Gamma}$, dual to $\mathrm{span}(C)$, with the cone over Y_{inv} in $\check{\mathbb{P}}^{10}_{\mathbb{C}}$, namely $\pi^{-1}(Y_{\mathrm{inv}})$. If $\dim \mathrm{span}(C) = 6$, then $\dim \widetilde{\Gamma} = 3$. Since X_{inv} is contained in the hyperplane $m_{11} + m_{22} + m_{33} + h = 0$, it follows that $\widetilde{\Gamma}$ passes through the point p_e with coordinates $l = -2$, $z_{11} = z_{22} = z_{33} = 1$ and all other coordinates being zero. Borel's construction can be rephrased as simply choosing a 3-space passing through p_e and intersecting with the cone over Y_{inv}. This cone has degree 10 and codimension 3 in $\check{\mathbb{P}}^{10}_{\mathbb{C}}$, so generically there are 10 solutions (possibly complex), each corresponding to a pair of legs. For a general 3-space $\widetilde{\Gamma}$ passing through p_e, one can ask three questions on reality:

(1) How many of the 10 points of $\widetilde{\Gamma} \cap \pi^{-1}(Y_{\mathrm{inv}})$ are real?
(2) How many of the real points above have real preimages under α? Namely, how many real legs does the curve C admit?
(3) Does the curve $X_{\mathrm{inv}} \cap \Lambda$ have real components, where Λ is the dual to $\widetilde{\Gamma}$ under SBSC?

The answers to Question (1) and (2) only depend on the projection of $\widetilde{\Gamma}$ to $\check{\mathbb{P}}^9_{\mathbb{C}}$. In order to obtain positive answers for Question 3, it is also convenient to start with the projection to $\check{\mathbb{P}}^9_{\mathbb{C}}$.

Definition 4.1. A *Borel subspace* Γ is a 3-space in $\check{\mathbb{P}}^9_{\mathbb{C}}$ passing through $\pi(p_e)$.

The following proposition settles Question (3).

Proposition 4.2. *Let Γ be a Borel subspace. Then there exists a 3-space $\widetilde{\Gamma}$ passing through p_e such that $\pi(\widetilde{\Gamma}) = \Gamma$ and $X_{\mathrm{inv}} \cap \Lambda$ has real components, where Λ is the dual of $\widetilde{\Gamma}$ under* SBSC.

Proof. Let $f\colon S \dashrightarrow \mathbb{P}^4_{\mathbb{C}}$ be the projection from the linear subspace U dual to $\pi^{-1}(\Gamma)$. Note that U is contained in the hyperplane $m_{11} + m_{22} + m_{33} + h = 0$. Hence

Table 1. Points in $\widetilde{\Gamma} \cap \pi^{-1}(Y_{\mathrm{inv}})$.

no. of real points	2	4	6	8	10
frequency	22	1067	3638	4035	1238

Table 2. Points in $\alpha^{-1}\big(\widetilde{\Gamma} \cap \pi^{-1}(Y_{\mathrm{inv}})\big)$.

no. of real points	2	4	6	8	10	12	14	16	18	20
frequency	0	4107	0	5240	0	650	0	3	0	0

the image of X_{inv} under f is contained in a linear 3-space, and $f|_{X_{\mathrm{inv}}}$ has one-dimensional fibers. Since X_{inv} has real components, it follows that there exist fibers $(f|_{X_{\mathrm{inv}}})^{-1}(q)$ with real components, for some $q \in \mathbb{P}^4_{\mathbb{C}}$. We just need to choose $\widetilde{\Gamma}$ dual to $f^{-1}(q)$; then $X_{\mathrm{inv}} \cap \Lambda$ coincides with $(f|_{X_{\mathrm{inv}}})^{-1}(q)$ and therefore has real components. $\square$

In order to get some statistical data on the answers to Question (1) and (2), we tested 10000 random examples[3] of Borel subspaces. The results are shown in Tables 1 and 2.

As one can see, the experimental data do not reveal any example of pods with 20 real legs. This is, however, misleading; see the next section.

4.2. **Icosapods via spectrahedra.** We conclude our work by showing how it is possible to construct a mobile icosapod with 20 real legs using some result in convex algebraic geometry.

Consider a 4-dimensional vector space A of symmetric 4×4-matrices over $\mathbb{R}$. Classically, the *spectrahedron* defined by A is the subset of A comprised of positive semidefinite matrices. One can also consider the spectrahedron as a subset of the projective space $\mathbb{P}(A) \cong \mathbb{P}^3$. The boundary of the spectrahedron consists of the semidefinite matrices with determinant 0, and hence its Zariski closure is a quartic surface in $\mathbb{P}^3$, called the *symmetroid* defined by A. In general, a symmetroid has 10 double points, corresponding to matrices of rank 2.

Given a spectrahedron whose symmetroid has 10 complex double points, its *type* is the pair of integers (a, b), where a is the number of real double points of the symmetroid and b is the number of real double points of the symmetroid that are also contained in the spectrahedron.

Theorem 4.3. *There is a bijective correspondence between quartic spectrahedra containing the matrix* $E := \left(\begin{smallmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{smallmatrix}\right)$ *and Borel subspaces. For a spectrahedron defined by a vector space A and the corresponding Borel subspace Γ, the following statement holds: if the spectrahedron has type (a, b), then Γ intersects Y_{inv} in a real points, and $a - b$ of them have real preimages under α.*

Proof. We identify $\check{\mathbb{P}}^9_{\mathbb{C}}$ with the projectivization of the vector space of symmetric 4×4 matrices in the following way: a point with homogeneous coordinates $z_{00}, \ldots, s_{23}$ corresponds to the class of the matrix

$$\begin{pmatrix} 2\,z_{00} & s_{01} & s_{02} & s_{03} \\ s_{01} & 2\,z_{11} & s_{12} & s_{13} \\ s_{02} & s_{12} & 2\,z_{22} & s_{23} \\ s_{03} & s_{13} & s_{23} & 2\,z_{33} \end{pmatrix}.$$

[3]The Maple code used to perform such experiments can be downloaded from `http://matteogallet.altervista.org/main/papers/icosapods2015/Icosapods.mpl`

A linear subspace A of dimension 3 in the space of symmetric matrices containing the matrix E corresponds then to a Borel subspace Γ.

The subvariety Y_{inv} corresponds to the subvariety of matrices of rank 2. A real matrix of rank 2 does not lie on the spectrahedron if and only if the quadratic form defined by it is a product of two distinct real linear forms, and this is true if and only if its preimage under α is real. $\square$

Degtyarev and Itenberg in [48] determined all possible types of quartic spectrahedra. In particular, spectrahedra of type $(10,0)$ do exist, hence by Theorem 4.3 they provide Borel subspaces intersecting Y_{inv} in 10 real points, each of them having two real preimages under the map α. This implies that there exist Borel icosapods with 20 real legs.

In [3], the authors give explicit examples of spectrahedra for all possible types. The given example of type $(10,0)$ does not contain the matrix E, but it is easy to adapt their example to one of the same type that does contain E.

4.3. **Example.** Starting from [3, Section 2, Case (10, 0)] we computed[4] the following example, which is suitable for graphical representation:

$$
\begin{aligned}
P_1 = p_4 &= \left(-\tfrac{19493}{142100}, -\tfrac{2088}{94325}, -\tfrac{24}{9625}\right), & p_1 = P_4 &= \left(-\tfrac{36411}{267844}, -\tfrac{1608}{177793}, \tfrac{504}{25399}\right), \\
P_2 = p_5 &= \left(-\tfrac{269}{5000}, \tfrac{39}{1000}, \tfrac{17}{500}\right), & p_2 = P_5 &= \left(-\tfrac{47}{368}, -\tfrac{12}{1771}, \tfrac{21}{1265}\right), \\
P_3 = p_6 &= \left(-\tfrac{1863}{14645}, -\tfrac{106851}{1555400}, \tfrac{2509}{222200}\right), & p_3 = P_6 &= \left(-\tfrac{15185}{112462}, -\tfrac{120}{149303}, \tfrac{48}{3047}\right).
\end{aligned}
$$

We apply a half-turn to the platform about a line ℓ through the point $(-\frac{1}{10}, 0, 0)$ in direction $(1, \frac{7003716944}{10000000000}, \frac{8}{10})$. In the resulting initial position, which is illustrated in Figure 1, the squared leg lengths of the first six legs read as follows:

$$
\begin{aligned}
d_1^2 = d_4^2 &= \tfrac{1081643179736912972309543483891375692}{276669953748621822688942197018838171875}, \\
d_2^2 = d_5^2 &= \tfrac{219482305781081742844809989061}{29002829339836395492656900000000}, \\
d_3^2 = d_6^2 &= \tfrac{4185335506762812187908674782558830797}{636621874987061375644008358435317156000}.
\end{aligned}
$$

For this input data the self-motion consists of two components. The trajectories of the component which passes through the initial position are illustrated in Figures 3 and 4. In the latter figure also the associated basic surface is displayed. An animation of this line-symmetric self-motion can be downloaded from `www.geometrie.tuwien.ac.at/nawratil/icosapod.gif`.

We close the paper by mentioning two open questions that we find of interest:

- Starting from spectrahedra of type (a,b) with $a-b \geq 4$ one may construct mobile pods with 16, 12 or 8 legs: is it true that a general mobile pod with 16, 12 or 8 legs is line-symmetric? (It is known that a, b have to be even numbers.)
- Identify all cases where more than 20 points move on spheres during a line-symmetric motion; i.e. (a) 1-dim, (b) 2-dim or even (c) 3-dim set of points with spherical trajectories. Case (c) is completely known due to Bricard [12], but cases (a) and (b) are still open. Examples for both cases are known (cf. Section 1.2.2 and [11, 12, 14, 18]).

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[4] The Maple code containing a similar computation can be downloaded from `http://matteogallet.altervista.org/main/papers/icosapods2015/Icosapods.mpl`

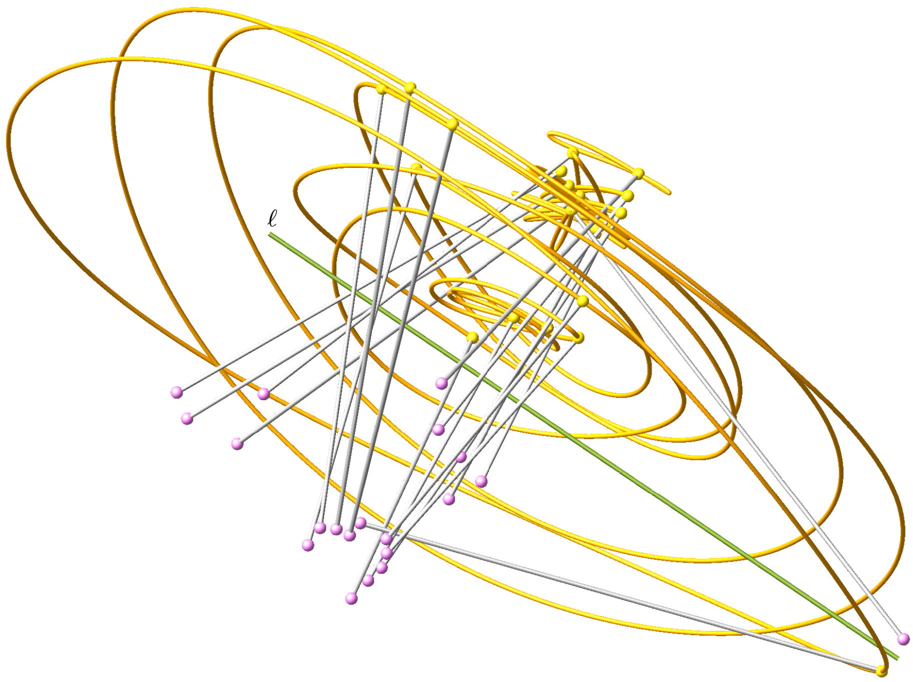

FIGURE 3. The 20 spherical trajectories passing through the initial position of the icosapod.

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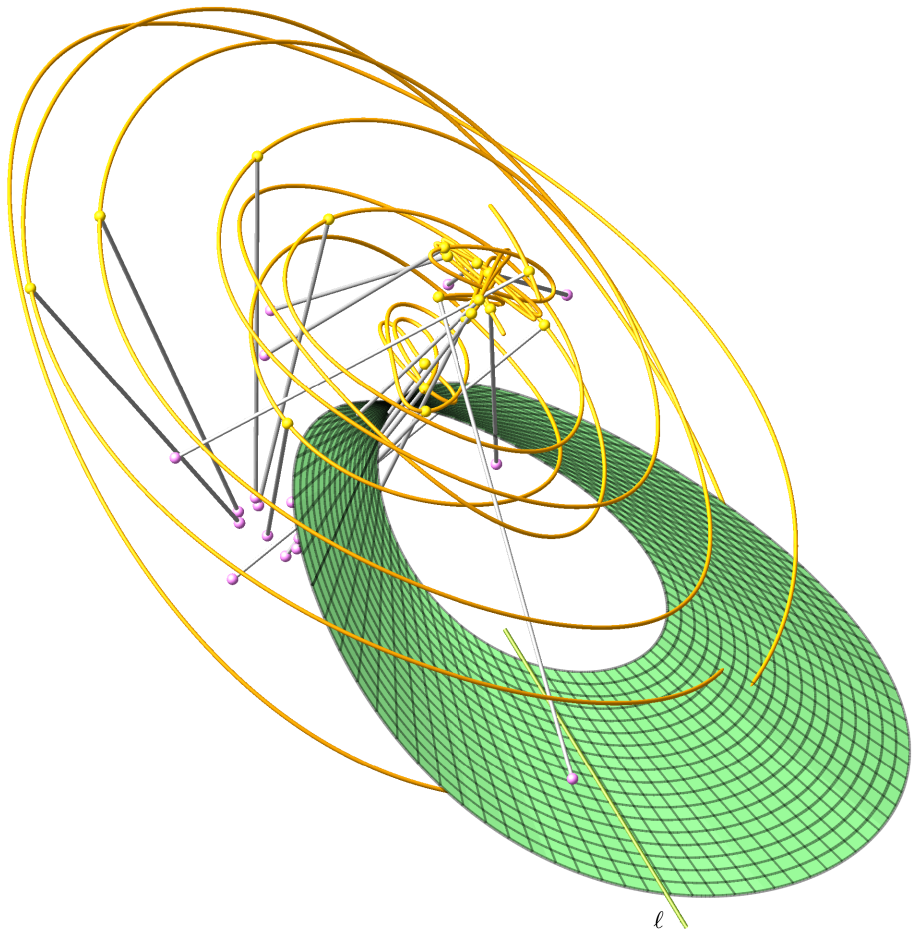

FIGURE 4. The same set-up as in Figure 3 but from another perspective. In addition a strip of the base surface of this line-symmetric self-motion is illustrated, where ℓ is highlighted. Note that one of the two displayed families of curves on the surface is composed of straight lines, i.e. the generators of the ruled surface.

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