Projective and affine symmetries and equivalences of rational curves in arbitrary dimension

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Abstract

We present a new algorithm to decide whether two rational parametric curves are related by a projective transformation and detect all such projective equivalences. Given two rational curves, we derive a system of polynomial equations whose solutions define linear rational transformations of the parameter domain, such that each transformation corresponds to a projective equivalence between the two curves. The corresponding projective mapping is then found by solving a small linear system of equations. Furthermore we investigate the special cases of detecting affine equivalences and symmetries as well as polynomial input curves. The performance of the method is demonstrated by several numerical examples.

Keywords: projective equivalences, affine equivalence, symmetry detection, rational curve, homogeneous polynomials

1. Introduction

It has been observed that many of the results from algebraic geometry and symbolic computation, which are available for rational curves (see the monograph by Sendra et al., 2008), are directly useful for addressing application-oriented problems in geometric modelling. In particular, several algorithms relying on symbolic computation have been designed that can solve specific problems associated with the design and the analysis of shapes.

The investigation of *singularities*, which is one of the classical topics in algebraic geometry, is clearly important for geometric design. For instance, several authors designed methods for detecting singularities of rational planar curves (Chen et al.,

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2008; Pérez-Díaz, 2007) as well as of rational space curves (Shi et al., 2013; Shi and Chen, 2010; Rubio et al., 2009) from their parametric representations, using resultants and μ-bases.

Another interesting question concerning planar rational curves is the computation and the study of their *offsets*. Some properties of offsets, such as the degree and the topological behaviour, have been investigated in Segundo and Sendra (2009) and Alcázar (2008).

Another line of research is devoted to *properties of the parameterization*. Among numerous papers on this topic, Pérez-Díaz (2006) addressed the issue of reparameterization to generate proper parameterizations, and Tabera (2011) explored the generation of optimal parameterizations in the sense of minimal coefficients.

The transformation between an *implicit and a parametric representation* of the curve is one of the most basic questions in algebraic geometry and therefore investigated in numerous publications (Rubio et al., 2006; Sederberg et al., 1997; Chen and Wang, 2002, to quote only a few).

The problem of *detecting symmetries and equivalences of curves* attracted substantial attention since it is an essential problem in Pattern Recognition, Computer Graphics and Computer Vision. For instance it is used to identify a given object with objects in a database. In Computer Graphics the knowledge about symmetries helps analyzing pictures and is applied to compression or shape completion.

Several papers on the detection of symmetries and equivalences of curves exist. In one of the earliest publications on this topic, Huang and Cohen (1996) examined affine transformations for classifying silhouettes of aircrafts, where they used B-spline moments which they approximated from a sample of points. Braß and Knauer (2004) investigated Euclidean symmetries of discrete 3-dimensional objects and proposed to apply their method to the control polygon of Bézier curves and surfaces. Lebmeir and Richter-Gebert (2008) and Lebmeir (2009) looked at Euclidean symmetries of algebraic curves given in implicit form.

More recently Alcázar (2014) and Alcázar et al. (2014a,b, 2015a,b) published a series of papers investigating the problem of symmetry and equivalence detection with respect to similarity transformations for parametric rational curves. They use the fact that the symmetry of a curve in proper parameterization is related to a rational linear transformation in the parameter domain. However, they do not consider general affine or projective equivalences. Sánchez-Reyes (2015) proposed a method for Euclidean symmetry detection of polynomial Bézier curves, based on using the Bernstein basis. For further references on symmetry detection see the introduction of Alcázar et al. (2014b).

Our paper is devoted to the detection of equivalences and symmetries of rational curves with respect to the group of *projective transformations*, of which affine ones are special cases. Similar to the works of Alcázar et al., we are looking for an exact and not approximated representation of the equivalences. We assume that exact input data is given and therefore we are able to use methods from the field of symbolic computation. More precisely, we use the monomial representations and homogeneous coordinates to derive algebraic equations that characterize the linear rational reparameterization connecting two equivalent curves.

Our method is more general than the existing ones since it handles all equivalences with respect to the full group of projective transformations and works for an arbitrary space dimension. In addition, it is easy to implement and provides good computational results for moderate degree.

The remainder of the paper is organized as follows. First we recall some basic geometric ideas and define our notation in Section 2. The main part of the paper is Section 3 where we derive a polynomial system whose solutions specify projective equivalences, characterize the size of the system and provide a comparison to a more naive approach. In Section 4 we consider some special cases, i.e., we are looking for affine equivalences and investigate the simplifications if the input curve is polynomial. Section 5 provides examples that show the simplicity of our method and give numerous further computational results. Finally in Section 6 we conclude this paper and describe some planned future work.

2. Preliminaries

We recall the different types of coordinates and recall the notions of projectively and affinely equivalent curves.

2.1. Coordinates

Throughout the paper we consider the field of real numbers, i.e., all coefficients of the curves and all variables describing the transformations and reparameterization are given as real numbers. We consider curves in the Euclidean space $\bar{E}^d$, which has been projectively closed (indicated by the bar) by adding points at infinity. Its points are represented by homogeneous coordinate vectors $\mathbf{x} = (x_0, x_1, \ldots, x_d)^T \in P^d(\mathbb{R}) = \mathbb{R}^{d+1}\backslash\{(0, 0, \ldots, 0)\}$. Linearly dependent pairs of homogeneous coordinate vectors represent the same point, and this relation will be denoted by $\simeq$. More precisely, we write $\mathbf{x} \simeq \mathbf{y}$ if and only if there exists $\mu \neq 0$ such that $\mathbf{x} = \mu\mathbf{y}$.

Homogeneous coordinate vectors with $x_0 = 0$ represent points at infinity, and the collection of these points forms the hyperplane at infinity. All other points can be represented by Cartesian coordinates $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_d)^T = (x_1/x_0, \ldots, x_d/x_0)^T$.

2.2. Rational curves and projective equivalences

Throughout the paper we consider two parametric rational curves $\mathcal{C}$ and $\mathcal{C}' \subset \bar{E}^d$, which are considered as point sets. Both curves are given by proper parameterizations

$$\begin{aligned} \mathbf{p} : P^1(\mathbb{R}) \to \mathcal{C} \subset \bar{E}^d, \quad & \mathbf{t} \mapsto \mathbf{p}(\mathbf{t}) = (p_0(t_0,t_1), p_1(t_0,t_1), \ldots, p_d(t_0,t_1)), \\ \mathbf{p}' : P^1(\mathbb{R}) \to \mathcal{C}' \subset \bar{E}^d, \quad & \mathbf{t} \mapsto \mathbf{p}'(\mathbf{t}) = (p'_0(t_0,t_1), p'_1(t_0,t_1), \ldots, p'_d(t_0,t_1)) \end{aligned}$$

with the parameter $\mathbf{t} = (t_0, t_1)$. Note that the prime symbol $'$ does not denote a differentiation but is used instead to distinguish between the two curves.

The domain of both parameterizations is the real projective line $P^1(\mathbb{R})$. Consequently, the homogeneous coordinates of both curves are homogeneous polynomials of degree n,

$$p_i(\mathbf{t}) = \sum_{j=0}^{n} c_{j,i} t_0^{n-j} t_1^j \quad \text{and} \quad p'_i(\mathbf{t}) = \sum_{j=0}^{n} c'_{j,i} t_0^{n-j} t_1^j, \quad i = 0, \ldots, d,$$

with coefficient vectors

$$\mathbf{c}_j = (c_{j,0}, c_{j,1}, \ldots, c_{j,d})^T \quad \text{and} \quad \mathbf{c}'_j = (c'_{j,0}, c'_{j,1}, \ldots, c'_{j,d})^T. \tag{1}$$

Curves given in standard (i.e., non-homogeneous) form are homogenized by replacing

$$t^j \quad \text{with} \quad t_0^{n-j} t_1^j.$$

Note that every rational curve, which is not given by a proper parameterization, can be reparameterized to obtain a proper one. The proof for planar curves, which is given by Sendra et al. (2008), applies to any space dimension d.

Furthermore we assume that the parameterizations are in reduced form, i.e.

$$\gcd(p_0(\mathbf{t}), p_1(\mathbf{t}), \ldots, p_d(\mathbf{t})) = \gcd(p'_0(\mathbf{t}), p'_1(\mathbf{t}), \ldots, p'_d(\mathbf{t})) = 1 \tag{2}$$

and of common degree n. Indeed, regular projective transformations preserve the degree of a curve, hence the degrees of projectively equivalent curves have to be equal. In particular, this implies that both curves possess the same degree

$$\begin{aligned} \max(\deg_{t_i}(p_0(\mathbf{t})), \deg_{t_i}(p_1(\mathbf{t})), \ldots, \deg_{t_i}(p_d(\mathbf{t}))) = n, \\ \max(\deg_{t_i}(p'_0(\mathbf{t})), \deg_{t_i}(p'_1(\mathbf{t})), \ldots, \deg_{t_i}(p'_d(\mathbf{t}))) = n, \end{aligned} \tag{3}$$

with respect to t_i, $i = 0, 1$.

We will assume that neither of the two curves is contained in a hyperplane. Consequently, the matrices (c_{ij}) resp. (c'_{ij}) formed by the coefficient vectors have rank $d + 1$. Clearly, this is only possible if the degree satisfies $n \geq d$.

We start with a simple technical lemma.

Lemma 1. *Two rational parameterizations* $\mathbf{p}(\mathbf{t})$ *and* $\mathbf{p}'(\mathbf{t})$ *in reduced form are equivalent, i.e.* $\mathbf{p}(\mathbf{t}) \simeq \mathbf{p}'(\mathbf{t})$ *holds for all* $\mathbf{t} \in P^1(\mathbb{R})$*, if and only if there exists a non-zero constant* μ *such that*

$$\mathbf{c}_j = \mu \mathbf{c}'_j, \quad j = 0, \ldots, n.$$

Proof. The equivalence of the two curves implies that there exists a rational function

$$\mu(\mathbf{t}) = \frac{\mu_1(\mathbf{t})}{\mu_0(\mathbf{t})} = \frac{p'_0(\mathbf{t})}{p_0(\mathbf{t})} = \frac{p'_1(\mathbf{t})}{p_1(\mathbf{t})} = \cdots = \frac{p'_d(\mathbf{t})}{p_d(\mathbf{t})}$$

where μ_0 are μ_1 are relatively prime polynomials, such that $\mathbf{p}(\mathbf{t}) = \mu(\mathbf{t})\mathbf{p}'(\mathbf{t})$. Consequently, the two rational curves satisfy

$$\mu_0(\mathbf{t})\mathbf{p}(\mathbf{t}) = \mu_1(\mathbf{t})\mathbf{p}'(\mathbf{t}).$$

This function is indeed a constant since

$$\mu_0 | \underbrace{\mathrm{gcd}(p'_0, p'_1, \ldots, p'_d)}_{=1} \quad \text{and} \quad \mu_1 | \underbrace{\mathrm{gcd}(p_0, p_1, \ldots, p_d)}_{=1}.$$

□

Recall that using homogeneous coordinates allows to represent any regular projective transformation f by a matrix multiplication

$$f : \bar{E}^d \to \bar{E}^d : \mathbf{x} \mapsto f(\mathbf{x}) = M\mathbf{x},$$

where $M = (m_{ij})_{i,j=0,\ldots,d}$ is a non-singular real matrix. If

$$m_{00} \neq 0 \quad \text{and} \quad m_{01} = \cdots = m_{0d} = 0, \tag{4}$$

then f is an affine transformation. For $d = 2$ the class of affine transformations includes translations, rotations, uniform and non-uniform scaling, reflections and shears. Projective transformations further include transformations that do not necessarily preserve parallel lines but collinearity and incidence. We consider pairs of curves which are related by regular affine and projective transformations.

Definition 2. Two curves $\mathcal{C}$ and $\mathcal{C}'$ are said to be *projectively (affinely) equivalent* if there exists a regular projective (affine) transformation f such that $\mathcal{C}' = f(\mathcal{C})$. Furthermore, $\mathcal{C}$ is said to possess a *projective (affine) symmetry* if there exists a regular projective (affine) transformation f, different from the identity, such that $\mathcal{C} = f(\mathcal{C})$.

	$\mathcal{C}' = f(\mathcal{C})$	$\mathcal{C} = f(\mathcal{C}), \quad f \neq \mathrm{id}$
M regular	$\mathcal{C}$ and $\mathcal{C}'$ are projectively equivalent	$\mathcal{C}$ has the projective symmetry f
A regular	$\mathcal{C}$ and $\mathcal{C}'$ are affinely equivalent	$\mathcal{C}$ has the affine symmetry f
$A^T A = \lambda I$	$\mathcal{C}$ and $\mathcal{C}'$ are similar	$\mathcal{C}$ has the self-similarity f
$A^T A = I$	$\mathcal{C}$ and $\mathcal{C}'$ are congruent	$\mathcal{C}$ has the symmetry f

Table 1: Projective equivalence of curves and its special cases

If $\mathcal{C}'$ is projectively equivalent to $\mathcal{C}$, then $\mathcal{C}$ is also projectively equivalent to $\mathcal{C}'$, as the projective transformation f is assumed to be regular. Moreover, each curve is projectively equivalent to itself by the identity map. The transitivity of the relation is implied by the group structure of regular projective mappings. Therefore, the projective equivalence defines an equivalence relation. The same argumentation holds for affine equivalences as well.

Two affinely equivalent curves are said to be *congruent* if the matrix

$$A = \left(\frac{m_{ij}}{m_{00}}\right)_{i,j=1,\ldots,d},$$

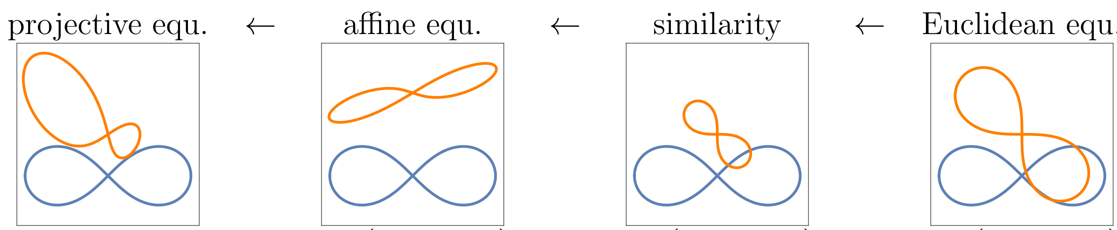
which is defined by the affine transformation, is orthogonal. Similarly, an affine symmetry is simply called a *symmetry* in this case (see, e.g., Alcázar et al., 2014b). Moreover, the two curves are said to be *similar* if the matrix A is a multiple of an orthogonal matrix, i.e., $A^T A = \lambda I$ with $\lambda \in \mathbb{R}^+$ (see, e.g., Alcázar et al., 2014a).

Self-similarities of rational curves are always symmetries (see e.g. Theorem 6 in Alcázar et al., 2015a), while non-rational curves (such as the logarithmic spiral) may possess more general self-similarities. Finally we note that the image of a curve with symmetries under a general affine (resp. projective) transformation has affine (resp. projective) symmetries, which are not symmetries.

Table 1 summarizes the different notions of equivalences and symmetries, whereas Figure 1 provides a graphical interpretation. The arrow "$\leftarrow$" in this figure indicates that projective equivalences are the most general among these different types of equivalence relations.

3. Detecting projective equivalences

We present a method to analyze whether two curves $\mathcal{C}, \mathcal{C}' \subset \bar{E}^d$ are projectively equivalent and to find all equivalences. This includes the construction of the associated projective transformations. We assume that the degrees satisfy $n > d$ since any two curves of degree d, which are not contained in hyperplanes, are related by infinitely many projective transformations.



M is regular	$M = \begin{pmatrix} m_{00} & 0 \\ b & m_{00}A \end{pmatrix}$, A is regular (nonuniform scaling, shear)	$M = \begin{pmatrix} m_{00} & 0 \\ b & m_{00}A \end{pmatrix}$, $A^T A = \lambda I$ (uniform scaling)	$M = \begin{pmatrix} m_{00} & 0 \\ b & m_{00}A \end{pmatrix}$, $A^T A = I$ (rotations, translations, reflections)

Figure 1: Dependencies between the different types of equivalence relations

3.1. The direct method

Recall that any two proper parameterizations of a rational curve are related by a linear rational reparameterization (Sendra et al., 2008), which is simply a regular projective transformation of the real projective line

$$\mathbf{r}(\mathbf{t}) = \underbrace{\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}}_{=\alpha} \mathbf{t} = \begin{pmatrix} \alpha_{00}t_0 + \alpha_{01}t_1 \\ \alpha_{10}t_0 + \alpha_{11}t_1 \end{pmatrix}$$

described by a regular matrix α. We investigate the transformation of the coefficients which is caused by such a reparameterization.

Lemma 3. *The reparameterized curve $\hat{\mathbf{p}} = \mathbf{p} \circ \mathbf{r}$,*

$$\mathbf{p}(\mathbf{r}(\mathbf{t})) = \hat{\mathbf{p}}(\mathbf{t}) = \sum_{j=0}^{n} \hat{\mathbf{c}}_j t_0^{n-j} t_1^j$$

has the coefficients

$$\hat{\mathbf{c}}_j(\alpha) = \sum_{i=0}^{n} \mathbf{c}_i \sum_{\ell=0}^{j} \binom{n-i}{\ell} \binom{i}{j-\ell} \alpha_{00}^{n-i-\ell} \alpha_{01}^{\ell} \alpha_{10}^{i-j+\ell} \alpha_{11}^{j-\ell} \tag{5}$$

for $j = 0, \ldots, n$.

Proof. A short computation gives

$$
\begin{aligned}
\mathbf{p}(\mathbf{r}(\mathbf{t})) &= \sum_{i=0}^{n} \mathbf{c}_i (\alpha_{00} t_0 + \alpha_{01} t_1)^{n-i} (\alpha_{10} t_0 + \alpha_{11} t_1)^i \\
&= \sum_{i=0}^{n} \mathbf{c}_i \left(\sum_{\ell=0}^{n-i} \binom{n-i}{\ell} \alpha_{00}^{n-i-\ell} t_0^{n-i-\ell} \alpha_{01}^{\ell} t_1^{\ell} \right) \left(\sum_{m=0}^{i} \binom{i}{m} \alpha_{10}^{i-m} t_0^{i-m} \alpha_{11}^{m} t_1^{m} \right) \\
&= \sum_{i=0}^{n} \mathbf{c}_i \sum_{\ell=0}^{n-i} \sum_{m=0}^{i} \binom{n-i}{\ell} \binom{i}{m} \alpha_{00}^{n-i-\ell} \alpha_{01}^{\ell} \alpha_{10}^{i-m} \alpha_{11}^{m} t_0^{n-m-\ell} t_1^{m+\ell} \\
&= \sum_{i=0}^{n} \mathbf{c}_i \sum_{j=0}^{n} \sum_{\ell+m=j} \binom{n-i}{\ell} \binom{i}{m} \alpha_{00}^{n-i-\ell} \alpha_{01}^{\ell} \alpha_{10}^{i-m} \alpha_{11}^{m} t_0^{n-j} t_1^{j} \\
&= \sum_{j=0}^{n} t_0^{n-j} t_1^{j} \sum_{i=0}^{n} \mathbf{c}_i \sum_{\ell+m=j} \binom{n-i}{\ell} \binom{i}{m} \alpha_{00}^{n-i-\ell} \alpha_{01}^{\ell} \alpha_{10}^{i-m} \alpha_{11}^{m}
\end{aligned}
$$

Comparing the coefficients confirms (5). □

Remark 4. For singular matrices α, the projective mapping $\mathbf{r}$ transforms the entire line into a single point, hence the coefficients of the reparameterized curve $\hat{\mathbf{p}} = \mathbf{p} \circ \mathbf{r}$ are all linearly dependent.

We identify projective equivalences by analyzing whether the coefficients are related by a projective transformation.

Proposition 5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be rational curves of degree $n > d$ with proper parameterizations $\mathbf{p}(\mathbf{t})$ and $\mathbf{p}'(\mathbf{t})$ satisfying (2) and (3). The two curves are projectively equivalent if and only if there exist a regular projective transformation matrix M and a projective transformation α of the real line, such that the coefficients of both curves satisfy*

$$
M\mathbf{c}'_j = \hat{\mathbf{c}}_j(\alpha), \quad j = 0, \ldots, n, \tag{6}
$$

see (5).

Proof. On the one hand, the conditions (6) imply that the two curves are projectively equivalent. On the other hand, we consider two projectively equivalent curves $\mathcal{C}'$ and $\mathcal{C}$. There exists a projective transformation f with the matrix M such that

$$
f(\mathcal{C}') = \mathcal{C}.
$$

We define $\mathbf{z}(\mathbf{t}) = M\mathbf{p}'(\mathbf{t})$. Consequently $\mathbf{z}(\mathbf{t})$ and $\mathbf{p}(\mathbf{t})$ are two proper parameterizations of the same curve $\mathcal{C}$. According to Lemma 4.17 of Sendra et al. (2008) there is a linear rational reparameterization $\mathbf{r}(\mathbf{t})$ – and hence an associated projective transformation α – such that

$$\mathbf{z}(\mathbf{t}) \simeq \mathbf{p}(\mathbf{r}(\mathbf{t})).$$

Thus we obtain that

$$\sum_{j=0}^{n} M\mathbf{c}'_j t_0^{n-j} t_1^j = M\mathbf{p}'(\mathbf{t}) = \mathbf{z}(\mathbf{t}) \simeq \mathbf{p}(\mathbf{r}(\mathbf{t})) = \hat{\mathbf{p}}(\mathbf{t}) = \sum_{j=0}^{n} \hat{\mathbf{c}}_j(\alpha) t_0^{n-j} t_1^j$$

where we use – from left to right – the representation of $\mathbf{p}'$, the definition of $\mathbf{z}(t)$, Lemma 4.17 of Sendra et al. (2008), the definition of $\hat{\mathbf{p}}$ and Lemma 3. We complete the proof by comparing the leftmost and rightmost terms and noting that Lemma 1 implies (6). Note that the constant μ of the homogeneous coordinates can be put into M. $\square$

Recall that we assume that neither of the two curves is contained in a hyperplane. Consequently, the regularity of M can be guaranteed by the regularity of α, cf. Remark 4. Thus, in addition to (6) we have

$$\det \alpha \neq 0.$$

In order to avoid the inequality constraint we replace it with the equation

$$(\det \alpha) u = 1 \tag{7}$$

which involves the additional variable u. Moreover, without loss of generality, it can be assumed that this additional variable satisfies the normalization condition

$$|u| = 1, \tag{8}$$

as the representation of the projective transformation α is only determined up to a constant factor. Nevertheless we keep the variable u in Equation (7) since this allows us to avoid the computation of several Gröbner bases.

Summing up, we arrive at a simple method for testing whether two curves are projectively equivalent:

Corollary 6. *Under the assumptions of the previous proposition, the two rational curves $\mathcal{C}$ and $\mathcal{C}'$ are projectively equivalent if and only if there exist transformation matrices M and α and a constant u such that the equations* (6) *and* (7) *are satisfied.*

This corollary leads to a system of $(d+1)(n+1)+1$ polynomial equations for $(d+1)^2+5$ unknowns in M (containing $(d+1)^2$ unknowns), α (4 unknowns) and u. The equations from formula (6) are linear in M but of degree n with respect the elements of α. Solving this system will be called the

D*irect method for detecting* ***P****rojectively equivalent curves.* (DP)

3.2. Reducing the number of unknowns

We observe that the system given by Equation (6) has a special structure, i.e., it is linear in the unknowns describing M and the right hand side are homogeneous polynomials of degree n in α. We use this knowledge to reduce the number of unknowns before solving the system with the help of standard computer algebra systems.

Proposition 7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be as in Proposition 5. The coefficient matrix*

$$(c'_{ij})_{i=0,\dots,d,j=0,\dots,n} \tag{9}$$

has $d+1$ linearly independent column (coefficient) vectors $\mathbf{c}'_{j(0)},\dots,\mathbf{c}'_{j(d)}$ and its kernel is spanned by basis vectors

$$\mathbf{b}^k = (b^k_j)_{j=0,\dots,n}, \quad k = 1,\dots,n-d.$$

The two curves are projectively equivalent if and only if there is a transformation α of the projective line, such that the equations

$$\sum_{j=0}^{n} \hat{c}_{ij}(\alpha) b^k_j = 0, \quad i = 0,\dots,d, \quad k = 1,\dots,n-d \tag{10}$$

and

$$\det\left(\hat{\mathbf{c}}_{j(0)}(\alpha),\dots,\hat{\mathbf{c}}_{j(d)}(\alpha)\right) \neq 0, \tag{11}$$

are satisfied, where the coefficients $\hat{c}_{ij}(\alpha)$ are given in (5).

Proof. Clearly, the coefficient matrix has rank $d+1$, hence its kernel has dimension $n-d$. This confirms the existence of the linearly independent columns and kernel basis vectors.

We show that the two equations (10) and (11) are equivalent to the condition stated in Proposition 5. First we confirm that the latter condition implies the two equations. On the one hand, Equation (6) ensures that the kernel of the coefficient matrix (9) is contained in the kernel of the coefficient matrix

$$(\hat{c}_{ij}(\alpha))_{i=0,\dots,d,j=0,\dots,n} \tag{12}$$

of the reparameterized curve, thereby proving the first equation. The other one follows from the fact that M is regular by restricting Equation (6) to the selected $d+1$ linearly independent columns $\mathbf{c}'_{j(0)},\dots,\mathbf{c}'_{j(d)}$ of the coefficient matrix (9).

Second we prove the other implication. The first equation (10) guarantees that the kernel of the matrix (12) contains the kernel of (9). This implies that the space spanned by the row vectors of the matrix in (12) is contained in the space spanned by row vectors of (9), since these spaces are the orthogonal complements of the kernels. This proves the existence of the matrix M in (6). Its regularity is again implied by the fact that the two coefficient matrices have rank $d+1$, due to (11) and to the assumption on the linearly independent $d+1$ column vectors. $\square$

Remark 8. The non-zero term $(\alpha_{00}\alpha_{11} - \alpha_{10}\alpha_{01})^{\frac{d(d+1)}{2}}$ appears as a factor in the determinant in Equation (11). Furthermore, when using Gröbner basis or other methods for solving a system of polynomial equations, avoiding inequalities is an advantage. We therefore replace the inequality Equation (11) with

$$\frac{\det\left(\hat{\mathbf{c}}_{j^{(0)}}(\alpha), \ldots, \hat{\mathbf{c}}_{j^{(d)}}(\alpha)\right)}{(\alpha_{00}\alpha_{11} - \alpha_{10}\alpha_{01})^{\frac{d(d+1)}{2}}}\, u = 1 \tag{13}$$

by introducing one additional variable u. As the projective transformation α is only determined up to a constant factor, we fix this degree of freedom by a similar normalization as in DP, in particular, by setting

$$|u| = 1. \tag{14}$$

Proposition 7 and Remark 8 lead to a system of $(n-d)(d+1)+1$ polynomial equations for only five unknowns, u and the elements in α. All but one are of degree n, while the remaining Equation (13) has degree $(n-d)(d+1)+1$. Solving this system will be called the

$$\textit{\textbf{R}educed method for detecting \textbf{P}rojectively equivalent curves.} \tag{RP}$$

3.3. Computation of projective equivalences

For both methods DP and RP, we first compute the Gröbner bases of the systems formed by equations (6) and (7) and equations (10) and (13), respectively. Finally we substitute $u = \pm 1$ and compute the solutions.

The direct method returns both the 2×2 matrices α that specify the reparameterizations $\mathbf{r}$ and the associated projective transformations. In contrast to this, the method RP computes the 2×2 matrices α only. Once the reparameterizations have been found, the corresponding projective transformations M are obtained simply by solving the linear systems of equations

$$M\mathbf{c}'_{j^{(\ell)}} = \hat{\mathbf{c}}_{j^{(\ell)}}(\alpha), \quad \ell = 0, \ldots, d, \tag{15}$$

	# of unknowns	degree	# equations
DP	$(d+1)^2+5$	n	$n(d+1)+d+1$
		$n+1$	1
RP	5	n	$n(d+1)-d^2-d$
		$n(d+1)-d^2-d+1$	1

Table 2: Characteristics of the non-linear polynomial systems for detecting projective equivalences of rational degree n curves in d-dimensional space.

for the $(d+1)^2$ unknown elements of M. The computational effort is negligible compared to the overall computation time.

The specific type of the equivalence can be found by investigating the properties of the transformation matrix M. More precisely, it is an affine equivalence if the elements satisfy

$$m_{0i} = 0, \quad \text{for} \quad i = 1, \dots, d.$$

It is a similarity (or even a congruence transformation) if additionally the condition

$$A^T A = \lambda I \quad \text{with} \quad A = \left(\frac{m_{ij}}{m_{00}}\right)_{i,j=1,\dots,d}$$

is fulfilled, where I is the $d \times d$ identity matrix (and the factor even satisfies $\lambda = 1$ for congruence transformations).

When applied to pairs $(\mathcal{C}, \mathcal{C})$ of identical curves, each of the two methods allows us to identify all projective symmetries. Once again, this includes all affine or Euclidean symmetries, which are found by analyzing the properties of the corresponding transformation Matrix M, analogously to the discussion above.

Finally we compare the characteristics of the non-linear polynomial systems for the two different methods in Table 2. This is the most time consuming part of our algorithm. For the direct method DP, the unknowns include the elements of the matrix M, since it uses an all-at-once approach. For the reduced method RP the maximum degree of one equation is increased but both the number of unknowns and the number of equations are decreased by $(d+1)^2$.

4. Special Cases - affine equivalences and polynomial input curves

We take a closer look at two special cases.

4.1. Affine equivalences

As mentioned in the end of the previous section, we may identify affine equivalences by investigating the transformation matrix M. However, if we are interested exclusively in this special case it might be advantageous to take this into account beforehand, for instance, if one is interested in the Gröbner Basis of this system per se. We will see in Section 5 that for DP this modification also improves the speed of computation, whereas the reduced methods RA and RP show a similar behaviour.

Concerning the direct method for affine equivalences, Corollary 6 leads to a system of $(d+1)(n+1)+1$ polynomial equations for $d(d+1)+6$ unknowns in M (containing $d(d+1)+1$ unknowns), α (4 unknowns) and u. The equations are linear in M but of degree $n+1$ with respect the elements of α and u. Solving this system will be called the

D*irect method for detecting* ***A****ffinely equivalent curves.* (DA)

A similar modification can be derived for the reduced method RP as well. Instead of decreasing the number of unknowns, this increases the number of equations. More precisely, we obtain additional equations involving only the first homogeneous coordinate. These equations are of degree n:

Corollary 9. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two rational curves as in Proposition 7. The two curves $\mathcal{C}$ and $\mathcal{C}'$ are affinely equivalent if and only if there exist a projective transformation α, defining a reparameterization, and a constant ω such that the equations*

$$\omega c'_{0k} = \hat{c}_{0k}(\alpha), \quad k = 0, \dots, n, \tag{16}$$

$$\sum_{j=0}^{n} \hat{c}_{ij}(\alpha) b_j^k = 0, \quad i = 0, \dots, d, \quad k = 1, \dots, n-d, \tag{17}$$

$$\det\left(\hat{\mathbf{c}}_{j^{(0)}}(\alpha), \dots, \hat{\mathbf{c}}_{j^{(d)}}(\alpha)\right) \neq 0 \tag{18}$$

are satisfied.

Proof. On the one hand, if the two curves are affinely equivalent, then the equations are obviously satisfied with $\omega = m_{0,0}$, see Equation (4) and Proposition 5.

On the other hand, consider two curves satisfying the equations (16), (17) and (18). First we note that (18) implies $\omega \neq 0$. The two curves are projectively equivalent according to Proposition 7, hence there exists a projective transformation M with

$$M\mathbf{c}'_j = \hat{\mathbf{c}}_j(\alpha), \quad j = 0, \dots, n.$$

We will show that M represents an affine transformation.

Any point $\mathbf{x}'$ has a unique representation

$$\mathbf{x}' = \sum_{\ell=0}^{d} \xi_\ell \mathbf{c}'_{j^{(\ell)}}$$

with certain real coefficients ξ_ℓ, as the selected $d+1$ columns $\mathbf{c}'_{j(0)}, \ldots, \mathbf{c}'_{j(d)}$ are in general position. Its image under the projective transformation then satisfies

$$\mathbf{x} = M\mathbf{x}' = \sum_{j=0}^{d} \xi_\ell \hat{\mathbf{c}}_{j(\ell)}(\alpha),$$

due to the linearity of the transformation. We now use these two representations and the additional Equation (16) to derive the relation

$$x_0 = \omega x'_0$$

between the 0-th coordinates. Consequently, the projective transformation M is indeed even an affine transformation since it maps any point $\mathbf{x}'$ at infinity (where $x'_0 = 0$) to another point at infinity. □

Consequently, in addition to equations (10) and (13) we add $n+1$ polynomial equations (16), which have degree n with respect to the four scalar unknowns in α and which are linear with respect to an additional variable ω. Solving this system will be called the

***R**educed method for detecting **A**ffinely equivalent curves.* (RA)

Remark 10. The direct method DA can be applied even if degree and space dimension are equal, i.e. $n = d$. The reduced method RA, however, is not applicable since the normalization (13) fails in this case. Indeed, it takes the form $cu = 1$ for some constant c in this situation.

The methods DA and RA only give solutions such that $m_{0i} = 0$ for $i > 0$ in the transformation matrix M. In the case of the reduced method, M can again be computed by solving the linear system of equations

$$M\mathbf{c}'_{j(\ell)} = \hat{\mathbf{c}}_{j(\ell)}(\alpha), \quad \ell = 0, \ldots, d,$$

for the $d(d+1)+1$ unknown elements in M. The computational effort is negligible compared to the overall computation time.

Once more, the specific type of the equivalence can be identified by investigating the properties of the transformation matrix M. When applied to pairs $(\mathcal{C}, \mathcal{C})$ of identical curves, each of the two methods allows us to detect all affine symmetries, which are again Euclidean symmetries if $A^T A = I$.

Finally we again compare the characteristics of the non-linear polynomial systems for the two different methods in Table 3, similar to the previous one.

	# of unknowns	maximum degree	# equations
DA	$d(d+1)+6$	n	$n(d+1)+d+1$
		$n+1$	1
RA	6	n	$n(d+2)+1-d^2-d$
		$n(d+1)-d^2-d$	1

Table 3: Characteristics of the non-linear polynomial systems for detecting affine equivalences of rational degree n curves in d-dimensional space.

4.2. Affine equivalences of polynomial curves

In the case of two polynomial input curves $\mathcal{C}$ and $\mathcal{C}'$ the problem on hand simplifies as the reparameterization $\mathbf{r}(\mathbf{t})$ is no longer a linear rational function but becomes a linear transformation only.

Corollary 11. *In the situation of Proposition 5, the projective transformation α defines a linear parameter transformation (i.e. $\alpha_{01}=0$) and the matrix M satisfies (4) if $\mathcal{C}$ and $\mathcal{C}'$ are two affinely invariant polynomial curves.*

Proof. Firstly, it is obvious that the matrix M has the structure (4), since M is an affine transformation. Secondly, the coefficients of a polynomial curve $\mathcal{C}$ satisfy $c_{0j}=0$ for $j>0$, hence (5) gives the relation

$$\hat{c}_{0j}(\alpha)=c_{00}\binom{n}{j}\alpha_{00}^{n-j}\alpha_{01}^{j}.$$

Together with (4), the equations (6) then imply $\alpha_{01}=0$ since the coefficients of the other curve fulfil the equations $c'_{0j}=0$ for $j>0$ also. □

Consequently it suffices to consider only linear reparameterizations, where $\alpha_{00}=1$ and $\alpha_{01}=0$, when detecting affine equivalences of polynomial curves. This observation has several consequences:

- The formulas from Lemma 3 for representing the coefficients of a reparameterized curve simplify to

$$\hat{\mathbf{c}}_j(\alpha)=\sum_{i=0}^{n}\mathbf{c}_i\binom{i}{j}\alpha_{10}^{i-j}\alpha_{11}^{j} \tag{19}$$

- When applying RA, the equations (16) are automatically satisfied for $k>0$ and the remaining one determines the value of ω. Hence we only consider (10) and (13) as in RP, but with (19) instead of (5). The normalization condition (14) is replaced by choosing $\alpha_{00}=1$. Hence the system consists of $n(d+1)-d^2-d$ non-linear equations in the three unknowns u, α_{10} and α_{11}.

This observation cannot be extended to the case of projective equivalences. In fact, two projectively equivalent polynomial curves may be related by a linear rational reparameterization, which is not a linear parameter transformation, as we show by a simple example.

Example 12. We consider two cubic polynomial curves

$$\mathbf{p}(\mathbf{t}) = \begin{pmatrix} t_0^3 \\ 3t_0^2 t_1 + 3t_0 t_1^2 - 3t_1^3 \\ t_1^3 \end{pmatrix} \quad \text{and} \quad \mathbf{p}'(\mathbf{t}) = \begin{pmatrix} t_0^3 \\ -3t_0^3 + 3t_0^2 t_1 + 3t_0 t_1^2 \\ t_1^3 \end{pmatrix}$$

which are projectively equivalent as they are related for instance by

$$M\mathbf{p}'(\mathbf{t}) = \mathbf{p}(\alpha\mathbf{t}) \quad \text{with} \quad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Additionally, the first curve possesses six projective symmetries. One of them takes the form

$$\hat{M}\mathbf{p}(\mathbf{t}) = \mathbf{p}(\alpha\mathbf{t}) \quad \text{with} \quad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{M} = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}.$$

Two of the five remaining projective symmetries are affine symmetries (including the identity).

5. Examples

The main computational costs of the methods are caused by solving the algebraic systems presented in Sections 3 and 4. Several numerical and symbolic methods for solving algebraic systems exist. The generation of efficient solvers for polynomial systems is an interesting and wide field of research. However, a detailed discussion of these methods is beyond the scope of the paper. Instead we will rely on existing methods, which have been implemented in well-established computer algebra systems. More precisely, we will use Mathematica® Version 10 and Singular 4-0-2 (Decker et al., 2015) for our numerical examples.

In the remainder of this section we first consider projective equivalences in the next section. We create the polynomial system of RP for a cubic planar rational curve before we show our results for some more complex curves in 2D and 3D. Second, we address the computation of affine equivalences in Section 5.2 and show the computational results for this case. Finally we investigate polynomial curves.

5.1. *Projective Equivalences*

First we consider our methods for finding all projective equivalences, i.e. the methods DP and RP.

5.1.1. *Folium of Descartes - a cubic example*

We investigate the two rational curves

$$\mathbf{p}(\mathbf{t}) = \begin{pmatrix} t_0^3 + t_1^3 \\ 3t_0^2 t_1 \\ 3t_0 t_1^2 \end{pmatrix} \quad \text{and} \quad \mathbf{p}'(\mathbf{t}) = \begin{pmatrix} -208t_0^3 + 108t_0^2 t_1 - 18t_0 t_1^2 + t_1^3 \\ -200t_0^3 + 144t_0^2 t_1 - 30t_0 t_1^2 + 2t_1^3 \\ 144t_0^3 - 60t_0^2 t_1 + 6t_0 t_1^2 \end{pmatrix}. \tag{20}$$

for projective equivalences. The first one is the Folium of Descartes, since choosing $t_0 = 1$ and $t_1 = t$ transforms $\mathbf{p}(\mathbf{t})$ to the usual parameterization. The second curve was derived by applying a reparameterization and an affine transformation.

We omit the details of the direct method and show the equations generated by RP. The coefficient-matrices take the form

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \quad (c'_{ij}) = \begin{pmatrix} -208 & 108 & -18 & 1 \\ -200 & 144 & -30 & 2 \\ 144 & -60 & 6 & 0 \end{pmatrix}$$

and the first one leads to

$$(\hat{c}_{ij}(\alpha)) = \begin{pmatrix} \alpha_{00}^3 + \alpha_{10}^3 & 3\alpha_{00}^2\alpha_{01} + 3\alpha_{10}^2\alpha_{11} & 3\alpha_{00}\alpha_{01}^2 + 3\alpha_{10}\alpha_{11}^2 & \alpha_{01}^3 + \alpha_{11}^3 \\ 3\alpha_{00}^2\alpha_{10} & 6\alpha_{00}\alpha_{01}\alpha_{10} + 3\alpha_{00}^2\alpha_{11} & 3\alpha_{01}^2\alpha_{10} + 6\alpha_{00}\alpha_{01}\alpha_{11} & 3\alpha_{01}^2\alpha_{11} \\ 3\alpha_{00}\alpha_{10}^2 & 3\alpha_{01}\alpha_{10}^2 + 6\alpha_{00}\alpha_{10}\alpha_{11} & 6\alpha_{01}\alpha_{10}\alpha_{11} + 3\alpha_{00}\alpha_{11}^2 & 3\alpha_{01}\alpha_{11}^2 \end{pmatrix}.$$

The kernel of the second one consists of one vector

$$\mathbf{b}^1 = \begin{pmatrix} 1 & 6 & 36 & 208 \end{pmatrix}$$

and hence we have the following system

$$\begin{aligned}
0 &= \alpha_{00}^3 + 18\alpha_{00}^2\alpha_{01} + 108\alpha_{00}\alpha_{01}^2 + 208\alpha_{01}^3 + \alpha_{10}^3 + 18\alpha_{10}^2\alpha_{11} + 108\alpha_{10}\alpha_{11}^2 + 208\alpha_{11}^3 \\
0 &= 3\alpha_{00}^2\alpha_{10} + 36\alpha_{00}\alpha_{01}\alpha_{10} + 108\alpha_{01}^2\alpha_{10} + 18\alpha_{00}^2\alpha_{11} + 216\alpha_{00}\alpha_{01}\alpha_{11} + 624\alpha_{01}^2\alpha_{11} \\
0 &= 3\alpha_{00}\alpha_{10}^2 + 18\alpha_{01}\alpha_{10}^2 + 36\alpha_{00}\alpha_{10}\alpha_{11} + 216\alpha_{01}\alpha_{10}\alpha_{11} + 108\alpha_{00}\alpha_{11}^2 + 624\alpha_{01}\alpha_{11}^2 \\
1 &= 9u(\alpha_{00}^3 + \alpha_{10}^3) \\
\pm 1 &= u
\end{aligned}$$

which possesses the real solutions

$$\alpha_1 = \pm\frac{1}{\sqrt[3]{1872}} \begin{pmatrix} -6 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \pm\frac{1}{\sqrt[3]{1872}} \begin{pmatrix} 2 & 0 \\ -6 & 1 \end{pmatrix}.$$

No.	name of the curve	parametric representation
1	lemniscate	$t \mapsto \begin{pmatrix} 1+6t^2+t^4 \\ 1-t^4 \\ 2t-2t^3 \end{pmatrix}$
2	epitrochoid	$t \mapsto \begin{pmatrix} 7(t^2+1)^2 \\ 1+18t^2-7t^4 \\ 4t-20t^3 \end{pmatrix}$
3	3-leaf rose	$t \mapsto \begin{pmatrix} (t^2+1)^2 \\ t-3t^3 \\ 1-3t^2 \end{pmatrix}$
4	deltoid	$t \mapsto \begin{pmatrix} 12-24t+24t^2-12t^3+3t^4 \\ -4+16t-12t^2+4t^3-t^4 \\ -8+24t-24t^2+8t^3 \end{pmatrix}$
5	astroid	$t \mapsto \begin{pmatrix} 125+450t+690t^2+576t^3+276t^4+72t^5+8t^6 \\ -27-54t-36t^2-8t^3 \\ 64+288t+528t^2+504t^3+264t^4+72t^5+8t^6 \end{pmatrix}$
6	offset of a cardioid	$t \mapsto \begin{pmatrix} 15(6561+2916t^2+486t^4+36t^6+t^8) \\ -39366+61236t^2-31104t^3+3456t^5-756t^6+6t^8 \\ -18t(4374-1296t-1134t^2+864t^3-126t^4-16t^5+6t^6) \end{pmatrix}$
7	epitrochoid4	$t \mapsto \begin{pmatrix} 1+5t^2+10t^4+10t^6+5t^8+t^{10} \\ 3+105t^2-410t^4+410t^6-105t^8-3t^{10} \\ -10t+280t^3-444t^5+280t^7-10t^9 \end{pmatrix}$

Table 4: Parameterizations of the curves considered in Sections 5.1.2 and 5.2.1

Solving (15) for the positive solution gives the affine transformations

$$M_1 = -\frac{1}{1872}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{1872}\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

5.1.2. Further experiments for projective equivalences

For higher degrees the size of the system of equations grows and the computation becomes too large for showing it here in detail. Nevertheless, implemented with Mathematica and Singular we derived coherent results, which we present in this section.

We apply the different methods for detecting projective equivalences to the seven curves which are listed in Table 4. Additionally we tested our methods for some higher degree flower curves, which are given by

$$t \mapsto \begin{pmatrix} (1+2t+2t^2)^{n+1} \\ (2t+2t^2)s(t) \\ (1+2t)s(t) \end{pmatrix}, \quad \text{where} \quad s(t) = \sum_{k=0}^{n} \binom{2n}{2k}(-1)^k t^{2k}(1+t)^{2(n-k)} \tag{21}$$

with $n = 2, 4, 6$. These curves have $2n$ leaves, and similar curves were also considered by Alcázar et al. (2014a). We will refer to them as curves No. 8 to 10. All these curves are shown in Figure 2.

For creating the Gröbner basis of the polynomial system (6) and (7) for DP and (10) and (13) for RP, respectively, we use Singular, which is specialized on Gröbner bases and hence offers quite a fast implementation for computing them. We use the degree reverse lexicographical ordering.

We export the obtained Gröbner basis, as the Singular function solve() will not give a solution, if the ideal is not zero dimensional. Then we add the additional normalization equations (8) and (14), respectively. For solving this system we use the function Solve[] in Mathematica, which provides a convenient representation of the solutions, even if they consist of a family of solutions, which depend on one or more variables. Compared to the time for computing the Gröbner basis, the additional effort of solving the resulting system is negligible, in particular, for increasing degree n. Possible complex solutions are neglected in the end or can be avoided by using the option "Reals" in Solve[].

We investigate the curves with respect to projective symmetries by using twice the same input curve and additionally we are looking for projective equivalences of two different inputs. For this we use the representation from Table 4 and Equation (21) as first input curve $\mathbf{p}(\mathbf{t})$ and apply a reparameterization given by the matrix α and a projective transformation M to obtain the input curves $\mathbf{p}'(\mathbf{t})$:

$$\alpha = \begin{pmatrix} 0 & 1 \\ 6 & 8 \end{pmatrix}, \quad M = \begin{pmatrix} 22 & 1 & -1 \\ 15 & 12 & 5 \\ -10 & 0 & 10 \end{pmatrix}.$$

We show these curves in Figure 2. We summarize the specifications of the curves as well as the computation time for the different methods in Table 5. The last three rows of the table show results obtained by applying the method to three pairs of different curves (e.g. the lemniscate and the epitrochoid), where the second one was considered in its original form and after applying both a projective transformation and a reparameterization. As expected, no projective equivalences were found in those cases.

If the computation took longer than one day or ran out of memory we aborted the process (indicated by $> 10^5$). All computations were performed on an Intel Core i7 PC, with 3.4 GHz and 32 GB RAM.

Even for the planar case the projective coordinate based method RP provides a remarkable speed-up. Moreover, the direct method did not provide results for all test cases. These effects will be even stronger in higher dimensions.

5.1.3. Space curves

We also applied our methods to space curves of low degrees (see Table 6). Again we applied a reparameterization and a projective transformation to an input curve

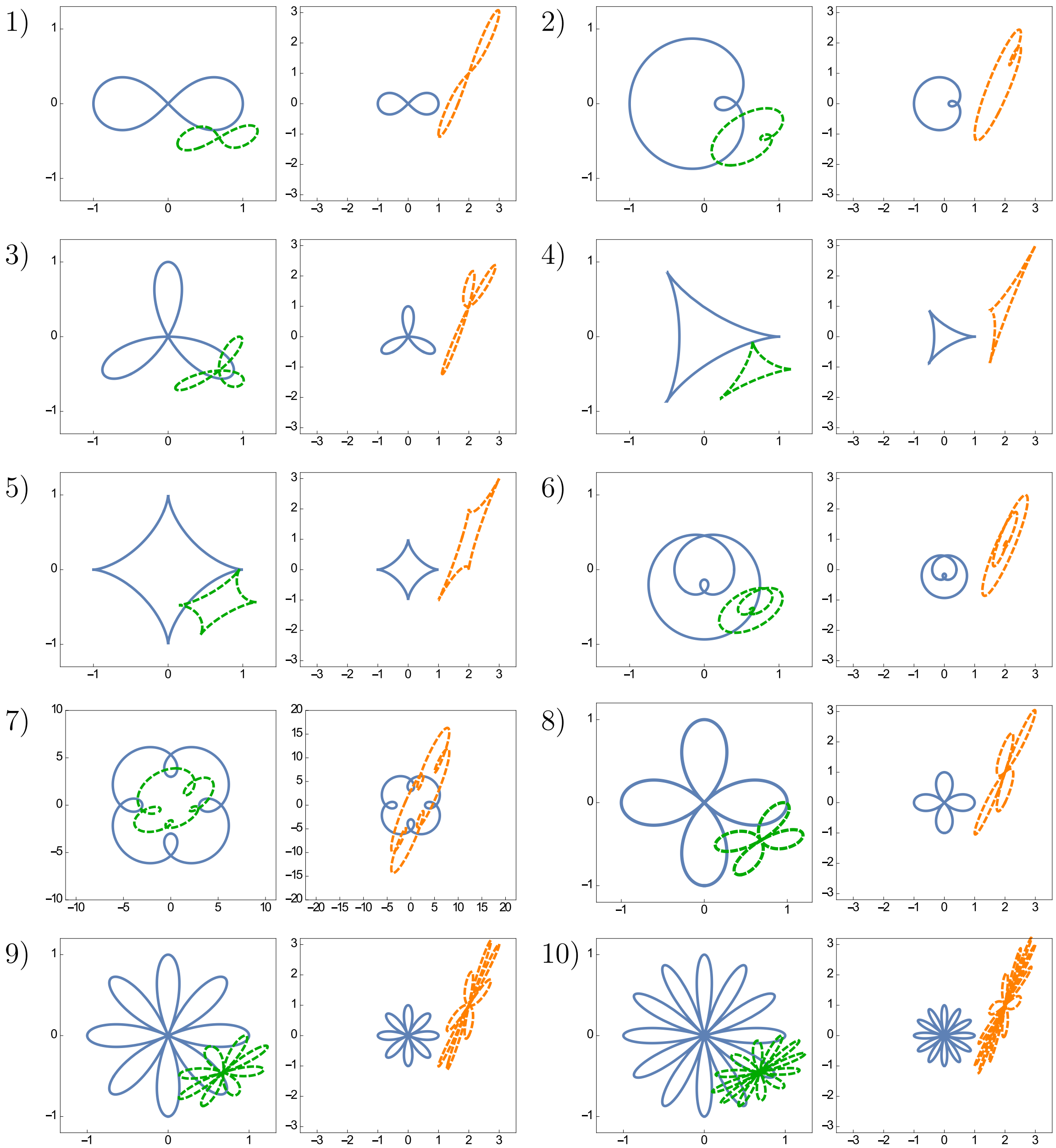


Figure 2: Examples 1-10: $\mathbf{p}(t)$ (solid blue), projectively transformed (dashed green, left), affinely transformed (dashed orange, right)

No.	deg.	# equiv.	Symmetries DP	Symmetries RP	Equivalences DP	Equivalences RP
1	4	4	9.8×10^{-1}	$< 10^{-2}$	1.4×10^{1}	3.0×10^{-2}
2	4	2	1.5×10^{0}	$< 10^{-2}$	9.6×10^{0}	$< 10^{-2}$
3	4	6	2.3×10^{-1}	1.0×10^{-2}	1.3×10^{1}	$< 10^{-2}$
4	4	6	2.2×10^{1}	1.0×10^{-2}	1.3×10^{1}	$< 10^{-2}$
5	6	8	1.0×10^{2}	3.0×10^{-2}	3.7×10^{2}	3.0×10^{-2}
6	8	2	7.8×10^{3}	4.6×10^{1}	9.3×10^{3}	4.6×10^{1}
7	10	8	8.1×10^{2}	2.5×10^{-1}	$> 10^{5}$	7.6×10^{-1}
8	6	8	1.6×10^{1}	7.0×10^{-2}	1.2×10^{2}	7.0×10^{-2}
9	10	16	6.5×10^{3}	2.3×10^{-1}	$> 10^{5}$	1.2×10^{0}
10	14	24	8.3×10^{3}	3.5×10^{2}	$> 10^{5}$	3.5×10^{2}
1+2	4	0	1.1×10^{0}	1.0×10^{-2}	9.4×10^{0}	1.0×10^{-2}
3+4	4	0	7.8×10^{0}	$< 10^{-2}$	5.6×10^{0}	$< 10^{-2}$
5+8	6	0	1.4×10^{1}	2.0×10^{-2}	1.0×10^{2}	3.0×10^{-2}

Table 5: Computation time in Singular for projective symmetries and equivalences of planar curves (time in seconds)

No.	name of the curve	parametric representation
11	3D Degree 4	$t \mapsto \begin{pmatrix} 1+t^4 \\ t+t^3 \\ t^3 \\ t^2 \end{pmatrix}$
12	3D Degree 6	$t \mapsto \begin{pmatrix} 125+450t+690t^2+576t^3+276t^4+72t^5+8t^6 \\ -27-54t-36t^2-8t^3 \\ 64+288t+528t^2+504t^3+264t^4+72t^5+8t^6 \\ 21+122t+216t^2+168t^3+60t^4+8t^5 \end{pmatrix}$
13	3D Degree 8	$t \mapsto \begin{pmatrix} 625+3000t+6400t^2+7920t^3+6216t^4+3168t^5+1024t^6+192t^7+16t^8 \\ -2027-8392t-14344t^2-12768t^3-5960t^4-1056t^5+224t^6+128t^7+16t^8 \\ 1664+7744t+16288t^2+20528t^3+17040t^4+9472t^5+3392t^6+704t^7+64t^8 \\ 405+1080t+1080t^2+480t^3+80t^4 \end{pmatrix}$

Table 6: Parameterizations of the curves considered in Section 5.1.3

$\mathbf{p}(\mathbf{t})$ in order to create some non-trivial input.

Table 7 suggests that the method works fine for low degree curves.

5.1.4. Randomly generated test cases for projective equivalences

For further experiments both in 2D and 3D we choose random coefficients of our curves as well as a random reparameterization and a random projective transformation. All these values are chosen as integer values with absolute value less then 10. Due to the choice of the input there is always at least one equivalence between the curves. For planar curves of higher degree then 7 and also lower degree 3D curves the direct method does not terminate within one day. With the method RP we can detect symmetries of random curves up to degree 9 within several minutes, see Table 8.

If we choose also the coefficients of the second curve randomly, the computation

No.	deg.	# equiv.	Symmetries DP	Symmetries RP	Equivalences DP	Equivalences RP
11	4	4	2.3×10^0	$< 10^{-2}$	2.3×10^2	$< 10^{-2}$
12	6	4	$> 10^5$	6.0×10^{-2}	$> 10^5$	2.0×10^{-2}
13	8	2	$> 10^5$	3.7×10^1	2.5×10^4	7.8×10^{-1}

Table 7: Computation time in Singular for projective symmetries and equivalences of space curves (time in seconds)

dim	deg.	Symmetry DP	Symmetry RP	Equivalences DP	Equivalences RP
2D	4	2.8×10^3	9.0×10^{-2}	1.0×10^4	2.7×10^{-1}
	5	7.8×10^3	1.2×10^0	7.6×10^3	1.4×10^0
	6	1.2×10^4	6.3×10^0	2.8×10^4	1.5×10^1
	7	2.9×10^4	4.2×10^1	3.5×10^4	1.2×10^2
	8	$> 10^5$	1.9×10^2	$> 10^5$	1.4×10^2
	9	$> 10^5$	6.2×10^2	$> 10^5$	2.2×10^3
3D	4	$> 10^5$	4.0×10^{-2}	$> 10^5$	4.0×10^{-1}
	5	$> 10^5$	1.0×10^0	$> 10^5$	1.6×10^0
	6	$> 10^5$	8.4×10^0	$> 10^5$	1.2×10^1
	7	$> 10^5$	3.7×10^1	$> 10^5$	8.6×10^1
	8	$> 10^5$	1.5×10^2	$> 10^5$	3.1×10^2
	9	$> 10^5$	6.7×10^2	$> 10^5$	1.7×10^3

Table 8: Computation time for projective equivalences with random curves (in seconds)

time remains within the same magnitude. As to be expected, no projective equivalences were found in these cases.

5.2. Affine Equivalences

In this section we present the results for the methods DA and RA designed to find affine equivalences and an example for affine equivalences of polynomial input curves.

5.2.1. Experiments for affine equivalences

Similar to Section 5.1.2 we applied a linear rational reparameterization and now an affine transformation on the input curves $\mathbf{p}(\mathbf{t})$ to obtain an input, that possesses affine equivalences. We summarize the computing time of all four methods DP, DA, RP and RA in Table 9. The reduced methods are still faster, although the additional assumptions improve the direct method significantly and concerning the computation time we cannot recognize an improvement in the reduced method. Nevertheless, as the methods RP and RA provide Gröbner bases of two different problems, the affine method might be useful in some applications. The computation time for affine Equivalences can be found in Table 9.

No.	deg.	# equiv.	Affine Equivalences DP	DA	RP	RA
1	4	4	1.1×10^1	4.0×10^{-2}	1.0×10^{-2}	$< 10^{-2}$
2	4	2	9.2×10^0	6.0×10^{-2}	$< 10^{-2}$	$< 10^{-2}$
3	4	6	1.2×10^1	1.0×10^{-2}	$< 10^{-2}$	$< 10^{-2}$
4	4	6	1.2×10^1	8.0×10^{-2}	$< 10^{-2}$	$< 10^{-2}$
5	6	8	2.3×10^4	4.4×10^0	1.6×10^{-1}	7.4×10^0
6	8	2	1.1×10^4	3.5×10^1	4.6×10^1	9.8×10^1
7	10	8	$> 10^5$	1.8×10^1	7.6×10^{-1}	2.6×10^{-1}
8	6	8	2.1×10^2	5.4×10^{-1}	7.0×10^{-2}	3.0×10^{-2}
9	10	16	$> 10^5$	1.4×10^1	1.4×10^0	2.7×10^{-1}
10	14	24	$> 10^5$	3.1×10^2	3.4×10^2	5.3×10^1
11	4	4	2.0×10^2	1.8×10^{-1}	$< 10^{-2}$	$< 10^{-2}$
12	6	4	$> 10^5$	7.1×10^0	2.0×10^{-2}	1.0×10^{-2}
13	8	2	$> 10^5$	2.7×10^1	3.2×10^0	2.9×10^0

Table 9: Computation time in Singular for affine equivalences of planar curves (time in seconds)

5.2.2. Polynomial curves

Finally we consider an example where the set of solutions is not a set of discrete values but is a family of solutions. We consider the two different parameterizations of a semi-cubical parabola

$$\mathbf{p}(\mathbf{t}) = \begin{pmatrix} t_0^3 \\ t_0 t_1^2 \\ t_1^3 \end{pmatrix} \quad \text{and} \quad \mathbf{p}'(\mathbf{t}) = \begin{pmatrix} t_0^3 \\ 2t_0 t_1^2 \\ t_1^3 \end{pmatrix}.$$

Hence we have the coefficient-matrices

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (c'_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which lead to

$$(\hat{c}_{ij}(\alpha)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\alpha_{10}^2 & 4\alpha_{10}\alpha_{11} & 2\alpha_{11}^2 & 0 \\ \alpha_{10}^3 & 3\alpha_{10}^2\alpha_{11} & 3\alpha_{10}\alpha_{11}^2 & \alpha_{11}^3 \end{pmatrix} \quad \text{and} \quad \mathbf{b}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

We note that $\mathbf{c}_0$, $\mathbf{c}_2$ and $\mathbf{c}_3$ are linearly independent and we obtain the polynomial system

$$0 = 4\alpha_{10}\alpha_{11}^2, \quad 0 = 3\alpha_{10}^2\alpha_{11}^2, \quad 1 = 2u\alpha_{11}^2.$$

Solving this leads to the reparameterization and transformation

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{11} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\alpha_{11}^2 & 0 \\ 0 & 0 & \alpha_{11}^3 \end{pmatrix}.$$

6. Conclusion

Two curves are projectively (resp. affinely) equivalent if there exists a regular projective (resp. affine) transformation, that maps one curve to the other. We investigated the case of rational curves using the fact that any two proper parameterizations of the same curve in reduced form are related by a linear rational reparameterization. This reparameterization can be represented by a projective transformation of the projective line describing the parameter domain.

Starting from the monomial coefficients we derived a system of polynomial equations whose solutions describe the projective (resp. affine) equivalences, i.e., the projective (resp. affine) transformations and the reparameterizations. The special structure of the system and using some basic linear algebra allowed us to obtain a compact form of these equations, i.e., we reduced the number of unknowns to five.

We implemented the algorithm using the CAS Mathematica and Singular and performed several computational experiments. For curves of moderate degrees (around 10 for planar curves), our implementation shows a similar performance as previous approaches (Alcázar, 2014; Alcázar et al., 2014a,b), i.e., it computes equivalences within a few seconds. However, these previous approaches were restricted to Euclidean equivalences and similarities, while we consider the more general case of projective (resp. affine) equivalences. Moreover, our approach works for arbitrary dimensions. To the best of our knowledge, this is the first work that uses the general concept of projective equivalences and considers an arbitrary space dimension of the embedding space.

Future work will be devoted to the extension to the surface case for triangular Bézier surfaces. Another possible future step is to investigate piecewise rational functions, as they are even more common in practical applications and they are often considered only for small degrees, where our method works fine.

Another challenging question consists in adapting the approach to numeric-symbolic computations, in order to deal with approximate symmetries and equivalences or with inexact input. Similar problems have been studied in the context of approximate algebraic geometry (see e.g. Pérez-Díaz et al., 2010, and the references therein). We expect that this question will be subject of future research.

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