Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations

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Abstract

We study the dimension and construct a basis for C^1-smooth isogeometric function spaces over two-patch domains. In this context, an isogeometric function is a function defined on a B-spline domain, whose graph surface also has a B-spline representation. We consider constructions along one interface between two patches. We restrict ourselves to a special case of planar B-spline patches of bidegree (p, p) with $p \geq 3$, so-called analysis-suitable G^1 geometries, which are derived from a specific geometric continuity condition. This class of two-patch geometries is exactly the one which allows, under certain additional assumptions, C^1 isogeometric spaces with optimal approximation properties (cf. [9]).

Such spaces are of interest when solving numerically fourth-order PDE problems, such as the biharmonic equation, using the isogeometric method. In particular, we analyze the dimension of the C^1-smooth isogeometric space and present an explicit representation for a basis of this space. Both the dimension of the space and the basis functions along the common interface depend on the considered two-patch parameterization. Such an explicit, geometry dependent basis construction is important for an efficient implementation of the isogeometric method. The stability of the constructed basis is numerically confirmed for an example configuration.

Keywords: isogeometric analysis, analysis-suitable G^1 geometries, C^1 smooth isogeometric functions, geometric continuity

1. Introduction

The problems discussed in this paper are inspired by isogeometric analysis (IGA), which was developed in [15]. The core idea of isogeometric analysis is to use the spline based representation of CAD models directly for the numerical analysis of partial differential equations (PDE). For a more detailed description of the isogeometric framework we refer to [3, 10].

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One of the advantages of IGA is the possibility to have discretization spaces of high order smoothness. These spaces can then be used to directly solve high order PDE problems. There exist several fourth (and higher) order problems of practical relevance. For their application in IGA see [2, 31], as well as [1, 4, 5, 19, 20] for Kirchhoff-Love shells, [12] for the Cahn-Hilliard equation and [13] for the Navier-Stokes-Korteweg equation.

The spline based representation of the physical domains allows for high order smoothness within one B-spline patch. However, most geometries of practical relevance cannot be represented directly with one patch but have to be parametrized using a multi-patch approach. It is not trivial to construct smooth function spaces over multi-patch domains. Many results for multi-patch domains can be derived from considerations on a single interface, hence in the following we will restrict ourselves to two-patch domains.

We are interested in C^1 isogeometric function spaces over two-patch domains. More precisely, we compute the dimension and construct a stable basis for a special class of, so called, analysis-suitable G^1 parameterizations. This class of geometries was first introduced in [9], where it was also shown that exactly the geometries of this class allow under certain assumptions C^1 isogeometric spaces with optimal approximation properties (cf. [9]). Note that an isogeometric function is C^1 if its graph surface is G^1. Hence, the analysis-suitability condition is a restriction of the more general geometric continuity condition. We refer to [28, 27] for the definition of geometric continuity.

The existing literature about the construction of C^1-smooth isogeometric functions on two-patch (and multi-patch) domains can be roughly classified into two possible approaches. The first one employs G^1-surface constructions around extraordinary vertices to obtain a set of C^1-smooth functions, see e.g. [18, 24, 25, 26]. In contrast, the second approach studies the entire space of C^1-smooth isogeometric functions on any given two-patch (or multi-patch) parameterization and generate a basis of the corresponding C^1 isogeometric space. Examples are [6, 9, 16, 17, 22, 23].

In this paper we follow the second approach, especially explored in [6, 16, 17, 22]. We extend these results in two main directions. First, these existing constructions and investigations are limited to piecewise bilinear domains. Our approach encloses the much wider class of analysis-suitable G^1 parameterizations, which contains the class of piecewise bilinear domains. Second, our construction works for non-uniform splines of arbitrary bidegree (p, p) with $p \geq 3$ and of arbitrary regularity r with $1 \leq r < p - 1$ within the two single patches. In contrast, the constructions [6, 22] are restricted to biquartic (for special cases) and to biquintic Bézier elements and the constructions [16, 17] are restricted to bicubic and biquartic uniform splines of regularity $r = 1$.

Further differences to [6, 22] are that our approach allows the construction of nested C^1 isogeometric spaces and that our basis functions are explicitly given, whereas the basis functions in [6, 22] are implicitly defined by means of minimal determining sets for the Bézier coefficients. Similar to [16, 17], the explicitly given basis functions possess a very small local support and are well conditioned. Moreover, the spline coefficients of our basis functions can be simply obtained by means of blossoming or fitting. This could provide a simple implementation in existing IGA libraries. In addition, we present the study of the dimension of the resulting C^1 isogeometric spaces for all possible configurations of analysis-suitable G^1 two-patch parameterizations.

The remainder of the paper is organized as follows. In Section 2 we describe some basic definitions and notations which are used throughout the paper. Section 3 recalls the concept of C^1-smooth isogeometric spaces over analysis-suitable G^1 two-patch parameterizations. The dimension of these spaces, which depends on the considered two-patch parameterization, is analyzed in Section 4. Then we present in Section 5 an explicit construction of basis functions, and describe in Section 6 their resulting spline coefficients by means of blossoming (and fitting). Finally, we conclude the paper in Section 7.

2. Preliminaries

Let ω be the interval $[0,1]$ or the unit square $[0,1]^2$. We denote by $\mathcal{S}(\mathcal{T}_k^{p,r},\omega)$ the (tensor-product) spline space of degree p (in each direction), which is defined on ω by choosing the open knot vector $\mathcal{T}_k^{p,r} = (t_0^{p,r},\ldots,t_{2p+1+k(p-r)}^{p,r})$ (in each direction) given by

$$\mathcal{T}_k^{p,r} = (\underbrace{0,\ldots,0}_{(p+1)-\text{times}}, \underbrace{\tau_1,\ldots,\tau_1}_{(p-r)-\text{times}}, \underbrace{\tau_2,\ldots,\tau_2}_{(p-r)-\text{times}},\ldots, \underbrace{\tau_k,\ldots,\tau_k}_{(p-r)-\text{times}}, \underbrace{1,\ldots,1}_{(p+1)-\text{times}}),$$

where $k \in \mathbb{N}_0$, $0 < \tau_i < \tau_{i+1} < 1$ for all $1 \leq i \leq k-1$ and k is the number of different inner knots (in each direction). Thereby r describes the resulting C^r-continuity of the space $\mathcal{S}(\mathcal{T}_k^{p,r},\omega)$ at all inner knots. The range for the regularity parameter r is in general $0 \leq r \leq p-1$. However, the focus here is on $r \geq 1$. Of course, in case of tensor-product splines, the knot vectors could be different in each direction. Moreover, the knot multiplicities could be different for every knot. To keep the presentation simple, we consider only the presented case. The spline spaces $\mathcal{S}(\mathcal{T}_k^{p,r},[0,1])$ and $\mathcal{S}(\mathcal{T}_k^{p,r},[0,1]^2)$ are spanned by the (tensor-product) B-splines $N_i^{p,r}$ and $N_{i,j}^{p,r} = N_i^{p,r} N_j^{p,r}$, $i,j = 0,\ldots,p+k(p-r)$, respectively. Each function $h \in \mathcal{S}(\mathcal{T}_k^{p,r},[0,1])$ and $z \in \mathcal{S}(\mathcal{T}_k^{p,r},[0,1]^2)$ possesses a B-spline representation

$$h(t) = \sum_{i=0}^{p+k(p-r)} d_i N_i^{p,r}(t) \tag{1}$$

and

$$z(u,v) = \sum_{i=0}^{p+k(p-r)} \sum_{j=0}^{p+k(p-r)} d_{i,j} N_{i,j}^{p,r}(u,v)$$

with spline control points $d_i \in \mathbb{R}$ and $d_{i,j} \in \mathbb{R}$, respectively.

In addition, we consider the knot vectors $\mathcal{T}_{k,\ell}^{p,r} = (t_0^{p,r},\ldots,t_{2p+2+k(p-r)}^{p,r})$ and $\mathcal{T}_{k,\ell,\ell'}^{p,r} = (t_0^{p,r},\ldots,t_{2p+3+k(p-r)}^{p,r})$, which are obtained by inserting into the knot vector $\mathcal{T}_k^{p,r}$ the knot τ_ℓ with the index $\ell \in \{1,\ldots,k\}$ (in case of $\mathcal{T}_{k,\ell}^{p,r}$) and the knots τ_ℓ and $\tau_{\ell'}$ with the indices $\ell,\ell' \in \{1,\ldots,k\}$ and $\ell \neq \ell'$ (in case of $\mathcal{T}_{k,\ell,\ell'}^{p,r}$). The resulting spline spaces are denoted by $\mathcal{S}(\mathcal{T}_{k,\ell}^{p,r},\omega)$ and $\mathcal{S}(\mathcal{T}_{k,\ell,\ell'}^{p,r},\omega)$, respectively, and are again C^r-smooth at all inner knots except at the knot τ_ℓ or at the knots τ_ℓ, $\tau_{\ell'}$, respectively, where the spaces are only C^{r-1}-smooth. Moreover, we denote by $\mathcal{P}^p(\omega)$ the space of (tensor-product) polynomials of degree p on ω.

3. $\boldsymbol{C}^1$-smooth isogeometric spaces and AS $\boldsymbol{G}^1$ two-patch geometries

We consider a planar domain $\Omega \subset \mathbb{R}^2$ composed of two quadrilateral spline patches $\Omega^{(L)}$ and $\Omega^{(R)}$, i.e. $\Omega = \Omega^{(L)} \cup \Omega^{(R)}$, which share a whole edge as common interface $\Gamma = \Omega^{(L)} \cap \Omega^{(R)}$. We assume that each patch $\Omega^{(S)}$, $S \in \{L, R\}$, is the image $\boldsymbol{F}^{(S)}([0,1]^2)$ of a regular, bijective geometry mapping

$$\boldsymbol{F}^{(S)} : [0,1]^2 \to \Omega^{(S)}, \quad \boldsymbol{F}^{(S)} \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2) \times \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2),$$

with the spline representations

$$\boldsymbol{F}^{(S)}(u,v) = \sum_{i=0}^{p+k(p-r)} \sum_{j=0}^{p+k(p-r)} \boldsymbol{c}_{i,j}^{(S)} N_{i,j}^{p,r}(u,v), \quad \boldsymbol{c}_{i,j}^{(S)} \in \mathbb{R}^2.$$

For the sake of simplicity, we assume that the two patches $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ share the common interface at

$$\boldsymbol{F}^{(L)}(0,v) = \boldsymbol{F}^{(R)}(0,v), \quad v \in [0,1].$$

We denote the parameterization of the common curve at Γ by $\boldsymbol{F}_0 : [0,1] \to \mathbb{R}^2$ and assume that $\boldsymbol{F}_0(v) = \boldsymbol{F}^{(L)}(0,v) = \boldsymbol{F}^{(R)}(0,v)$. The space of isogeometric functions on Ω is given as

$$\mathcal{V} = \{\phi : \Omega \to \mathbb{R} \text{ such that } \phi \circ \boldsymbol{F}^{(S)} \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2),\ S \in \{L, R\}\}.$$

The graph surface $\boldsymbol{\Sigma} \subset \Omega \times \mathbb{R}$ of an isogeometric function $\phi \in \mathcal{V}$ consists of the two graph surface patches

$$\boldsymbol{\Sigma}^{(S)} : [0,1]^2 \to \Omega^{(S)} \times \mathbb{R}, \quad S \in \{L, R\},$$

possessing the form

$$\boldsymbol{\Sigma}^{(S)}(u,v) = (\boldsymbol{F}^{(S)}(u,v), g^{(S)}(u,v))^T,$$

where $g^{(S)} = \phi \circ \boldsymbol{F}^{(S)} \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2)$. Since the geometry mappings $\boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, are given, an isogeometric function $\phi \in \mathcal{V}$ is determined by the two associated spline functions $g^{(S)}$, $S \in \{L, R\}$, with the spline representations

$$g^{(S)}(u,v) = \sum_{i=0}^{p+k(p-r)} \sum_{j=0}^{p+k(p-r)} d_{i,j}^{(S)} N_{i,j}^{p,r}(u,v), \quad d_{i,j}^{(S)} \in \mathbb{R}.$$

We are interested in C^1-smooth isogeometric functions $\phi \in \mathcal{V}$. We assume C^0-smoothness condition of ϕ, that results in

$$g^{(L)}(0,v) = g^{(R)}(0,v), \tag{2}$$

for $v \in [0,1]$. We denote the function along the common interface by $g(v) = g^{(L)}(0,v) = g^{(R)}(0,v)$. Let us consider the space $\mathcal{V}^1$ of C^1-smooth isogeometric functions on Ω, i.e.

$$\mathcal{V}^1 = \mathcal{V} \cap C^1(\Omega),$$

in more detail. An isogeometric function $\phi \in \mathcal{V}$ belongs to $\mathcal{V}^1$ if and only if the two graph surface patches $\mathbf{\Sigma}^{(L)}$ and $\mathbf{\Sigma}^{(R)}$ possess a well defined tangent plane along the common interface $\mathbf{\Sigma}^{(L)} \cap \mathbf{\Sigma}^{(R)}$, compare [9, 16, 17]. This is equivalent to the condition that there exist functions $\alpha^{(L)}, \alpha^{(R)}, \beta : [0,1] \rightarrow \mathbb{R}$ such that for all $v \in [0,1]$

$$\alpha^{(L)}(v)\alpha^{(R)}(v) < 0 \tag{3}$$

and

$$\alpha^{(R)}(v)D_u\mathbf{\Sigma}^{(L)}(0,v) - \alpha^{(L)}(v)D_u\mathbf{\Sigma}^{(R)}(0,v) + \beta(v)D_v\mathbf{\Sigma}^{(R)}(0,v) = \mathbf{0}, \tag{4}$$

see [28]. The above described equivalent conditions are called geometric continuity of order 1 or G^1-smoothness (cf. [14, 28]).

Note that the first two equations (4), i.e.

$$\alpha^{(R)}(v)D_u\boldsymbol{F}^{(L)}(0,v) - \alpha^{(L)}(v)D_u\boldsymbol{F}^{(R)}(0,v) + \beta(v)D_v\boldsymbol{F}_0(v) = \mathbf{0}, \tag{5}$$

uniquely determine the functions $\alpha^{(L)}$, $\alpha^{(R)}$ and β up to a common function $\gamma : [0,1] \rightarrow \mathbb{R}$ (with $\gamma(v) \neq 0$) by

$$\alpha^{(L)} = \gamma(v)\bar{\alpha}^{(L)}, \alpha^{(R)} = \gamma(v)\bar{\alpha}^{(R)} \text{ and } \beta(v) = \gamma(v)\bar{\beta}(v), \tag{6}$$

where

$$\bar{\alpha}^{(S)}(v) = \det(D_u\boldsymbol{F}^{(S)}(0,v), D_v\boldsymbol{F}_0(v)),\ S \in \{L,R\}, \tag{7}$$

and

$$\bar{\beta}(v) = \det(D_u\boldsymbol{F}^{(L)}(0,v), D_u\boldsymbol{F}^{(R)}(0,v)), \tag{8}$$

see e.g. [9, 28].

Therefore, an isogeometric function $\phi \in \mathcal{V}$ belongs to $\mathcal{V}^1$ if and only if the equation

$$\alpha^{(R)}(v)D_u g^{(L)}(0,v) - \alpha^{(L)}(v)D_u g^{(R)}(0,v) + \beta(v)D_v g(v) = 0 \tag{9}$$

is satisfied for all $v \in [0,1]$. Equations (2) and (9) lead to linear constraints on the spline control points $d_{i,j}^{(S)}$ of $g^{(S)}$, $S \in \{L,R\}$, with indices (S,i,j) belonging to the index space

$$I_\Gamma = \{(S,i,j) \mid S \in \{L,R\},\ i = 0,1 \text{ and } j = 0,\ldots,p+k(p-r)\}.$$

Moreover, we denote by I the index space formed by all spline control points $d_{i,j}^{(S)}$, $S \in \{L,R\}$, i.e.

$$I = \{(S,i,j) \mid S \in \{L,R\} \text{ and } i,j = 0,\ldots,p+k(p-r)\}.$$

Furthermore, for functions $\alpha^{(L)}, \alpha^{(R)}$ and β satisfying equations (5) there exist non-unique functions $\beta^{(L)}, \beta^{(R)} : [0,1] \rightarrow \mathbb{R}$ such that

$$\beta(v) = \alpha^{(L)}(v)\beta^{(R)}(v) - \alpha^{(R)}(v)\beta^{(L)}(v), \tag{10}$$

see e.g. [9, 28].

Note that for generic patches $\boldsymbol{F}^{(L)}, \boldsymbol{F}^{(R)} \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2) \times \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2)$ the functions $\alpha^{(L)}, \alpha^{(R)}$ and β fulfill $\alpha^{(L)}, \alpha^{(R)} \in \mathcal{S}(\mathcal{T}_k^{2p-1,r-1}, [0,1])$ as well as $\beta \in \mathcal{S}(\mathcal{T}_k^{2p,r}, [0,1])$. For special configurations the degree may be lower and the regularity may be higher.

Motivated by [9, 28], we restrict in the following the considered geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ to geometry mappings possessing linear functions $\alpha^{(L)}$, $\alpha^{(R)}$, $\beta^{(L)}$, $\beta^{(R)}$, as stated in the following definition.

Definition 1 (Analysis-suitable G^1, cf. [9])**.** The two-patch geometry parameterization $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ is analysis-suitable G^1 (AS G^1) if there exist $\alpha^{(L)}$, $\alpha^{(R)}$, $\beta^{(L)}$, $\beta^{(R)} \in \mathcal{P}^1([0,1])$, with $\alpha^{(L)}$ and $\alpha^{(R)}$ relatively prime (i.e., $\deg(\gcd(\alpha^{(L)}, \alpha^{(R)})) = 0$) satisfying equations (5) and (10).

The condition on $\deg(\gcd(\alpha^{(L)}, \alpha^{(R)})) = 0$ was not used in [9] but is not restrictive: if $\deg(\gcd(\alpha^L, \alpha^R)) = 1$ one can replace $\alpha^{(L)}$, $\alpha^{(R)}$ and β by $\alpha^{(L)}/\gcd(\alpha^{(L)}, \alpha^{(R)})$, $\alpha^{(R)}/\gcd(\alpha^{(L)}, \alpha^{(R)})$ and $\beta/\gcd(\alpha^{(L)}, \alpha^{(R)})$, respectively. Obviously, the polynomial β is divisible by $\gcd(\alpha^{(L)}, \alpha^{(R)})$, which can be seen best in (10). With $\deg(\gcd(\alpha^{(L)}, \alpha^{(R)})) = 0$, the functions $\alpha^{(L)}$, $\alpha^{(R)}$ and β are uniquely determined up to a common constant.

It was shown in [9] when the functions $\alpha^{(S)}$ and $\beta^{(S)}$, $S \in \{L, R\}$, are assumed to be of higher degree or even to be spline functions along the interface, then the polynomial representation along the interface is reduced to lower degrees. Hence, these spaces do not guarantee optimal approximation order. For this more general case the investigation of a dimension formula or of a basis construction should be possible in similar way as presented in the following sections, but is beyond the scope of this paper.

4. Dimension of the space $\mathcal{V}^1$ for AS G^1 two-patch geometries

We consider AS G^1 geometries $\boldsymbol{F}^{(L)}, \boldsymbol{F}^{(R)} \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2) \times \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2)$ with the corresponding functions $\alpha^{(L)}$, $\alpha^{(R)} \in \mathcal{P}^1([0,1])$ and $\beta \in \mathcal{P}^2([0,1])$.

Let d_α be the maximum degree of the functions $\alpha^{(S)}$, i.e.

$$d_\alpha = \max(\deg(\alpha^{(L)}), \deg(\alpha^{(R)})).$$

Here we consider the actual degree of the functions. Since $\alpha^{(L)}, \alpha^{(R)} \in \mathcal{P}^1([0,1])$, we obtain either $d_\alpha = 0$ or $d_\alpha = 1$. Let z_β be defined as the number of different inner knots where the function β possesses the value zero, i.e.

$$z_\beta = |\{v_0 \in \{\tau_1, \ldots, \tau_k\} \mid \beta(v_0) = 0\}|.$$

Since $\beta \in \mathcal{P}^2([0,1])$, $z_\beta \in \{0, 1, 2, k\}$.

Depending on β, we define a new knot vector $\widetilde{\mathcal{T}_k^p}$. First, if $\beta = 0$ then $\widetilde{\mathcal{T}_k^p} = \mathcal{T}_k^{p,r}$. Otherwise, assuming $\beta \neq 0$, we have three cases: if $z_\beta = 0$, we set $\widetilde{\mathcal{T}_k^p} = \mathcal{T}_k^{p,r+1}$, if $z_\beta = 1$ or $z_\beta = 2$, we set $\widetilde{\mathcal{T}_k^p} = \mathcal{T}_{k,\ell}^{p,r+1}$ or $\widetilde{\mathcal{T}_k^p} = \mathcal{T}_{k,\ell,\ell'}^{p,r+1}$, respectively, where $\ell, \in \{1, \ldots, k\}$, and possibly $\ell' \in \{1, \ldots, k\}$ with $\ell \neq \ell'$, are the indices of τ_ℓ and $\tau_{\ell'}$, which are roots of β.

We are interested in the dimension of the isogeometric space $\mathcal{V}^1$. Clearly, the space $\mathcal{V}^1$ is the direct sum of the two subspaces

$$\mathcal{V}_1^1 = \{\phi \in \mathcal{V}^1 \mid d_{i,j}^{(S)} = 0 \text{ for } S \in \{L, R\} \text{ and } (S, i, j) \in I_\Gamma\}$$

and

$$\mathcal{V}_2^1 = \{\phi \in \mathcal{V}^1 \mid d_{i,j}^{(S)} = 0 \text{ for } S \in \{L, R\} \text{ and } (S, i, j) \in I \setminus I_\Gamma\},$$

which implies that

$$\dim \mathcal{V}^1 = \dim \mathcal{V}_1^1 + \dim \mathcal{V}_2^1.$$

In [17], the functions of $\mathcal{V}_1^1$ and $\mathcal{V}_2^1$ have been called basis functions of the first and second kind, respectively. We first state the dimension of $\mathcal{V}_1^1$.

Lemma 2. *The dimension of $\mathcal{V}_1^1$ is equal to*

$$\dim V_1^1 = 2(p + k(p - r) - 1)(p + k(p - r) + 1).$$

Proof. Clearly, $\dim \mathcal{V}_1^1$ is equal to the number of control points $d_{i,j}^{(S)}$, $S \in \{L, R\}$, possessing indices $(S, i, j) \in I \setminus I_\Gamma$, i.e. $\dim \mathcal{V}_1^1 = |I \setminus I_\Gamma|$. $\square$

To analyze the dimension of $\mathcal{V}_2^1$, we need additional tools describing the situation at Γ. Some of these tools have been (similarly) introduced in [9]. Consider the transversal vector $\boldsymbol{d}^{(S)}$ defined on Γ such that

$$\boldsymbol{d}^{(S)} \circ \boldsymbol{F}_0(v) = (D_u \boldsymbol{F}^{(S)}(0, v), D_v \boldsymbol{F}_0(v))(1, -\beta^{(S)}(v))^T \frac{1}{\alpha^{(S)}(v)}, \; S \in \{L, R\}.$$

Observe that $\boldsymbol{d}^{(L)} = \boldsymbol{d}^{(R)}$ (which is equivalent to (5)) and therefore we simply set

$$\boldsymbol{d} = \boldsymbol{d}^{(L)} = \boldsymbol{d}^{(R)}.$$

In addition, we consider the space of traces and transversal derivatives on Γ and its pullback, which are given by

$$\mathcal{V}_\Gamma^1 = \{\Gamma \ni (x, y) \mapsto (\phi(x, y), \nabla \phi(x, y) \cdot \boldsymbol{d}(x, y)), \text{ such that } \phi \in \mathcal{V}_2^1\}$$

and

$$\widehat{\mathcal{V}_\Gamma^1} = \{(\phi, \nabla \phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0, \text{ such that } \phi \in \mathcal{V}_2^1\},$$

respectively. The transversal vector $\boldsymbol{d}$, $\mathcal{V}_\Gamma^1$ and $\widehat{\mathcal{V}_\Gamma^1}$ depend only on the choice of $\beta^{(L)}$ and $\beta^{(R)}$, since $\alpha^{(L)}$ and $\alpha^{(R)}$ are now uniquely determined by the geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$. Associated to $\boldsymbol{d}$, we consider the transversal derivative of ϕ with respect to $\boldsymbol{d}$ on Γ, i.e. $(\nabla \phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0$.

Clearly, for $\phi \in \mathcal{V}_2^1$ the function $\phi \circ \boldsymbol{F}_0$ is a spline function. More precisely, we have:

Lemma 3. *If $\phi \in \mathcal{V}_2^1$, then $\phi \circ \boldsymbol{F}_0 \in \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0, 1])$.*

Proof. Analyzing equation (9), we observe that $\phi \circ \boldsymbol{F}_0 \in \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1])$, since $D_u g^{(S)}(0,v) \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1])$. □

The following lemma ensures that for $\phi \in \mathcal{V}_2^1$ the function $(\nabla\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0$ is a spline function, too.

Lemma 4. *If $\phi \in \mathcal{V}_2^1$, then $(\nabla\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0 \in \mathcal{S}(\mathcal{T}_k^{p-d_\alpha, r-1}, [0,1])$.*

Proof. Recall that $\boldsymbol{d} \circ \boldsymbol{F}_0(v) = \boldsymbol{d}^{(L)} \circ \boldsymbol{F}_0(v) = \boldsymbol{d}^{(R)} \circ \boldsymbol{F}_0(v)$. An isogeometric function ϕ belong to the space $\mathcal{V}_2^1$ if and only if

$$(\nabla^{(L)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0(v) = (\nabla^{(R)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0(v) = (\nabla\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0(v)$$

for all $v \in [0,1]$. Since

$$(\nabla^{(S)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0(v) = \frac{D_u g^{(S)}(0,v) - \beta^{(S)}(v) D_v g(v)}{\alpha^{(S)}(v)}, \; S \in \{L, R\}, \tag{11}$$

we obtain that $\phi \in \mathcal{V}_2^1$ if and only if

$$(D_u g^{(L)}(0,v) - \beta^{(L)}(v) D_v g(v))\alpha^{(R)}(v) = (D_u g^{(R)}(0,v) - \beta^{(R)}(v) D_v g(v))\alpha^{(L)}(v) \tag{12}$$

for all $v \in [0,1]$. (Condition (12) is exactly the same as condition (9) by substituting β via (10).) Recall that $\deg(\gcd(\alpha^{(L)}, \alpha^{(R)})) = 0$. Therefore, by dividing Equation (12) by $\alpha^{(L)}$, we see that $(\nabla^{(L)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0 \in \mathcal{S}(\mathcal{T}_k^{p-\deg(\alpha^{(L)}), r-1}, [0,1])$. Analogously we can show that $(\nabla^{(R)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0 \in \mathcal{S}(\mathcal{T}_k^{p-\deg(\alpha^{(R)}), r-1}, [0,1])$ and obtain that $(\nabla\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0 \in \mathcal{S}(\mathcal{T}_k^{p-d_\alpha, r-1}, [0,1])$. □

There is a one-to-one correspondence between trace and transversal derivative at Γ, and $\phi \in \mathcal{V}_2^1$.

Proposition 5. *For any $(g_0, g_1) \in \widehat{\mathcal{V}_\Gamma^1}$ there exists a unique $\phi \in \mathcal{V}_2^1$ such that $(g_0, g_1) = (\phi, \nabla\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0$, given, for $S \in \{L, R\}$, by*

$$\phi \circ \boldsymbol{F}^{(S)} = g_0(v)(N_0^{p,r}(u) + N_1^{p,r}(u)) + \left(\alpha^{(S)}(v) g_1(v) + \beta^{(S)}(v) g_0'(v)\right) \frac{\tau_1}{p} N_1^{p,r}(u). \tag{13}$$

Proof. Recall the notation $\phi \circ \boldsymbol{F}^{(S)} = g^{(S)}(u,v)$. Equation (11) is equivalent to

$$D_u g^{(S)}(0,v) = \alpha^{(S)}(v)(\nabla^{(S)}\phi \cdot \boldsymbol{d}) \circ \boldsymbol{F}_0(v) + \beta^{(S)}(v) g_0'(v) = \alpha^{(S)}(v) g_1(v) + \beta^{(S)}(v) g_0'(v). \tag{14}$$

By Taylor expansion of $g^{(S)}(u,v)$ and using equation (14), we obtain

$$\begin{aligned} g^{(S)}(u,v) &= g^{(S)}(0,v) + (D_u g^{(S)})(0,v)u + \mathcal{O}(u^2) \\ &= g_0(v) + (\alpha^{(S)}(v) g_1(v) + \beta^{(S)}(v) g_0'(v))u + \mathcal{O}(u^2) \\ &= g_0(v)(N_0^{p,r}(u) + N_1^{p,r}(u)) + (\alpha^{(S)}(v) g_1(v) + \beta^{(S)}(v) g_0'(v)) \frac{\tau_1}{p} N_1^{p,r}(u), \end{aligned}$$

for $S \in \{L, R\}$. □

Two useful corollaries of Proposition 5 follow below. Both hold for any possible choice of $\beta^{(L)}$ and $\beta^{(R)}$.

Corollary 6. *The dimension of* $\mathcal{V}_2^1$ *is equal to the dimension of* $\widehat{\mathcal{V}_\Gamma^1}$.

Corollary 7. *It holds that* $(g_0, g_1) \in \widehat{\mathcal{V}_\Gamma^1}$ *if and only if:*

$$\begin{aligned} g_0 &\in \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1]) && (15) \\ g_1 &\in \mathcal{S}(\mathcal{T}_k^{p-d_\alpha, r-1}, [0,1]) && (16) \\ \alpha^{(S)} g_1 + \beta^{(S)} g_0' &\in \mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]), \ \textit{for } S \in \{L, R\}. && (17) \end{aligned}$$

Proof. The statement follows from Lemmas 3–4 and (13) in Proposition 5. □

To investigate the dimension of $\widehat{\mathcal{V}_\Gamma^1}$, we consider the spaces

$$\Gamma_0 = \{g_0 \text{ such that } (g_0, g_1) \in \widehat{\mathcal{V}_\Gamma^1} \text{ for some } g_1\} \equiv \{\phi \circ \boldsymbol{F}_0, \text{ such that } \phi \in \mathcal{V}_2^1\}$$

and

$$\Gamma_1 = \{(0, g_1) \in \widehat{\mathcal{V}_\Gamma^1}\}.$$

Clearly, $\dim \widehat{\mathcal{V}_\Gamma^1} = \dim \Gamma_0 + \dim \Gamma_1$. The following two lemmas state the dimension of Γ_0 and Γ_1.

Lemma 8. *It holds that*

$$\Gamma_0 = \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1]) \tag{18}$$

and consequently

$$\dim \Gamma_0 = p + k(p - r - 1) + 1 + z_\beta.$$

Proof. Thanks to Corollary 7, we need to show that for all $g_0 \in \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1])$ we can construct g_1 such that (16) and (17) holds.

In case of $\beta \equiv 0$, and also when $\beta \neq 0$ with $z_\beta = 0$, we can simply set $g_1(v) = 0$. In the rest of the proof, we always assume $\beta \neq 0$.

In case of $z_\beta = 1$, we choose $g_1(v) = -\frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)} g_0'(v) \in \mathcal{S}(\mathcal{T}_k^{p-1,r-1}, [0,1])$, where τ_ℓ is a root of β. We need to prove (17), that is, to show that

$$\left(\beta^{(S)} - \alpha^{(S)} \frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)} \right) g_0', \tag{19}$$

is C^r-smooth, for $S \in \{L, R\}$. We only need to check the r-th derivative of (19), for $S \in \{L, R\}$, which is

$$r \left(\beta^{(S)\prime}(v) - \alpha^{(S)\prime}(v) \frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)} \right) g_0^{(r)}(v) + \left(\beta^{(S)}(v) - \alpha^{(S)}(v) \frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)} \right) g_0^{(r+1)}(v),$$

that is continuous in $[0,1] \setminus \tau_\ell$ for the regularity of g_0 and continuous when $v \to \tau_\ell$ since the second addendum vanishes in the limit.

In case of $z_\beta = 2$, we choose $g_1(v) = -(\frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)}\hat{g}_0'(v) + \frac{\beta^{(L)}(\tau_{\ell'})}{\alpha^{(L)}(\tau_{\ell'})}\tilde{g}_0'(v)) \in \mathcal{S}(\mathcal{T}_k^{p-1,r-1}, [0,1])$, where $\hat{g}_0 \in \mathcal{S}(\mathcal{T}_{k,\ell}^{p,r+1}, [0,1])$ and $\tilde{g}_0 \in \mathcal{S}(\mathcal{T}_{k,\ell'}^{p,r+1}, [0,1])$ are non-unique functions such that $g_0 = \hat{g}_0 + \tilde{g}_0$, and $\tau_\ell, \tau_{\ell'}$ with the indices $\ell, \ell' \in \{1, \dots, k\}$ and $\ell \neq \ell'$ are the two roots of β. As before, we need to prove (17), that is, to show that

$$\left(\beta^{(S)} - \alpha^{(S)}\frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)}\right)\hat{g}_0' + \left(\beta^{(S)} - \alpha^{(S)}\frac{\beta^{(L)}(\tau_{\ell'})}{\alpha^{(L)}(\tau_{\ell'})}\right)\tilde{g}_0', \tag{20}$$

is C^r-smooth, for $S \in \{L, R\}$. We only need to check again that the r-th derivative of (20), for $S \in \{L, R\}$, is continuous in $[0,1]$, which can be done analogous to the case $z_\beta = 1$. The dimension of Γ_0 follows directly from the definition of the spline space. This concludes the proof. □

Lemma 9. *It holds that*

$$\Gamma_1 = \{0\} \times \mathcal{S}(\mathcal{T}_k^{p-d_\alpha,r}) \tag{21}$$

and consequently

$$\dim \Gamma_1 = p + k(p-r-1) + (1-d_\alpha)(k+1).$$

Proof. Thanks to Corollary 7, $(0, g_1) \in \Gamma_1$ if and only if $g_1 \in \mathcal{S}(\mathcal{T}_k^{p-d_\alpha,r})$. The dimension of Γ_1 follows directly from the definition of the spline space. □

This leads to the following result.

Lemma 10. *The dimension of $\mathcal{V}_2^1$ is equal to*

$$\dim \mathcal{V}_2^1 = 2(p + k(p-r-1)) + 1 + (1-d_\alpha)(k+1) + z_\beta.$$

Finally, we obtain:

Theorem 11. *The dimension of $\mathcal{V}^1$ is equal to*

$$\dim \mathcal{V}^1 = \underbrace{2(p+k(p-r)-1)(p+k(p-r)+1)}_{\dim \mathcal{V}_1^1} + \underbrace{2(p+k(p-r-1)) + 1 + (1-d_\alpha)(k+1) + z_\beta}_{\dim \mathcal{V}_2^1}.$$

Remark 12. Our results are in agreement with those in [17], where the special case of bilinear geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ with $d_\alpha = 1$, $z_\beta = 0$, $p = 3, 4$ and $r = 1$ was considered.

5. Basis of the space $\mathcal{V}^1$ for AS G^1 two-patch geometries

We present an explicit basis construction for the space $\mathcal{V}^1$ for AS G^1 two-patch geometries. Our basis will consist of a basis for the space $\mathcal{V}_1^1$ and of a basis for the space $\mathcal{V}_2^1$.

5.1. *Basis of* $\mathcal{V}_1^1$

The functions in $\mathcal{V}_1^1$ are not influenced by the interface Γ. Hence, a basis of $\mathcal{V}_1^1$ can be constructed in a straightforward way from the standard basis on a single patch. Consider for $i = 2, \ldots, p + k(p - r)$, $j = 0, \ldots, p + k(p - r)$, the isogeometric functions $\phi_{i,j}^{(L)}$ and $\phi_{i,j}^{(R)}$ determined by

$$\phi_{i,j}^{(L)} : \; g^{(L)}(u,v) = N_{i,j}^{p,r}(u,v), \; g^{(R)}(u,v) = 0 \text{ and } \phi_{i,j}^{(R)} : \; g^{(L)}(u,v) = 0, \; g^{(R)}(u,v) = N_{i,j}^{p,r}(u,v).$$

Then the collection of isogeometric functions $\{\phi_{i,j}^{(S)}\}_{(S,i,j)\in I\setminus I_\Gamma}$ forms a basis of the space $\mathcal{V}_1^1$. Note that these basis functions do not depend on the geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ (and therefore do not depend on $\alpha^{(L)}$, $\alpha^{(R)}$ and β, too). This is in contrast to the basis functions of $\mathcal{V}_2^1$, see Section 5.2.

5.2. *Basis of* $\mathcal{V}_2^1$

We present a construction of a basis of the space $\mathcal{V}_2^1$. Thereby, the resulting basis functions depend on $\alpha^{(L)}$, $\alpha^{(R)}$, $\beta^{(L)}$ and $\beta^{(R)}$, and hence on the geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$. The idea is to generate a basis of $\mathcal{V}_2^1$ by means of a basis of Γ_0 and of a basis of Γ_1 (technically by means of a basis of $\widehat{\mathcal{V}_\Gamma^1}$), since Proposition 5 provides an explicit representation for the desired basis functions of $\mathcal{V}_2^1$, given by

$$g^{(S)}(u,v) = g_0(v)(N_0^{p,r}(u) + N_1^{p,r}(u)) + \left(\alpha^{(S)}(v)g_1(v) + \beta^{(S)}(v)g_0'(v)\right)\frac{\tau_1}{p}N_1^{p,r}(u), \tag{22}$$

for $S \in \{L, R\}$. More precisely, the construction works as follows:

1. We select that pair of functions $\beta^{(L)}$ and $\beta^{(R)}$, such that (10) holds and which minimizes the term
$$||\beta^{(L)}||^2_{L_2([0,1])} + ||\beta^{(R)}||^2_{L_2([0,1])}. \tag{23}$$
In case of $\beta = 0$, we have $\beta^{(L)}(v) = \beta^{(R)}(v) = 0$.
2. Let $\tilde{n} = \dim \mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1])$. Depending on β, we first choose for the space $\mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1])$ a basis $\tilde{N}_i$, $i = 1, \ldots, \tilde{n}-1$, and then for each pair $(g_0, g_1) = (\tilde{N}_i, \tilde{g}_{1,i})$, $i = 0, \ldots, \tilde{n}-1$, the function $\tilde{g}_{1,i}$ as follows:
   - *Case* $\beta = 0$ *or* $z_\beta = 0$: The functions $\tilde{N}_i$, $i = 1, \ldots, \tilde{n} - 1$, are the B-splines of $\mathcal{S}(\widetilde{\mathcal{T}_k^p}, [0,1])$, and $\tilde{g}_{1,i}(v) = 0$.
   - *Case* $\beta \neq 0$ *and* $z_\beta = 1$: Let τ_ℓ with the index $\ell \in \{1, \ldots, k\}$ be the root of β. The functions $\tilde{N}_i$, $i = 1, \ldots, \tilde{n} - 2$, are the B-splines of $\mathcal{S}(\mathcal{T}_k^{p,r+1}, [0,1])$ and the function $\tilde{N}_{\tilde{n}-1}$ is one of the B-splines of $\mathcal{S}(\mathcal{T}_{k,\ell}^{p,r+1}, [0,1])$ with the property $\tilde{N}_{\tilde{n}-1}(\tau_\ell) \neq 0$. The function $\tilde{g}_{1,i}$ is given by
$$\tilde{g}_{1,i}(v) = \begin{cases} -\frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)}\tilde{N}_i'(v) & \text{if } i = \tilde{n} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

- *Case $\beta \neq 0$ and $z_\beta = 2$:* Let $\tau_\ell, \tau_{\ell'}$ with the indices $\ell, \ell' \in \{1, \dots, k\}$ and $\ell \neq \ell'$ be the two roots of β. The functions $\tilde{N}_i$, $i = 1, \dots, \tilde{n} - 3$, are the B-splines of $\mathcal{S}(\mathcal{T}_k^{p,r+1}, [0,1])$, the function $\tilde{N}_{\tilde{n}-2}$ is one of the B-splines of $\mathcal{S}(\mathcal{T}_{k,\ell}^{p,r+1}, [0,1])$ with the property $\tilde{N}_{\tilde{n}-2}(\tau_\ell) \neq 0$, and the function $\tilde{N}_{\tilde{n}-1}$ is one of the B-splines of $\mathcal{S}(\mathcal{T}_{k,\ell'}^{p,r+1}, [0,1])$ with the property $\tilde{N}_{\tilde{n}-1}(\tau_{\ell'}) \neq 0$. The function $\tilde{g}_{1,i}$ is given by

$$\tilde{g}_{1,i}(v) = \begin{cases} -\frac{\beta^{(L)}(\tau_\ell)}{\alpha^{(L)}(\tau_\ell)} \tilde{N}_i'(v) & \text{if } i = \tilde{n} - 2, \\ -\frac{\beta^{(L)}(\tau_{\ell'})}{\alpha^{(L)}(\tau_{\ell'})} \tilde{N}_i'(v) & \text{if } i = \tilde{n} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then each pair $(g_0, g_1) = (\tilde{N}_i, \tilde{g}_{1,i})$, $i = 0, \dots, \tilde{n} - 1$, determines a basis function $\phi_{0,i}$ via representation (22), i.e.

$$\phi_{0,i}: \; g^{(S)}(u,v) = \tilde{N}_i(v)(N_0^{p,r}(u) + N_1^{p,r}(u)) + (\alpha^{(S)}(v)\tilde{g}_{1,i}(v) + \beta^{(S)}(v)\tilde{N}_i'(v))\frac{\tau_1}{p} N_1^{p,r}(u),$$

for $S \in \{L, R\}$. In case of $\beta \neq 0$, the representation of $\phi_{0,i}$ simplifies to

$$\phi_{0,i}: \; g^{(S)}(u,v) = \tilde{N}_i(v)(N_0^{p,r}(u) + N_1^{p,r}(u)) + \beta^{(S)}(v)\tilde{N}_i'(v)\frac{\tau_1}{p} N_1^{p,r}(u), \; S \in \{L, R\}, \tag{24}$$

except for the index $i = \tilde{n} - 1$ if $z_\beta = 1$ and for the index $i = \tilde{n} - 2$ or $i = \tilde{n} - 1$ if $z_\beta = 2$. In case of $\beta = 0$, the representation of $\phi_{0,i}$ even simplifies to

$$\phi_{0,i}: \; g^{(S)}(u,v) = N_i^{p,r}(v)(N_0^{p,r}(u) + N_1^{p,r}(u)), \; S \in \{L, R\}.$$

3. Let $\bar{n} = \dim \mathcal{S}(\mathcal{T}_k^{p-d_\alpha,r}, [0,1])$, and let $\bar{N}_j$, $j = 0, \dots, \bar{n} - 1$, be the B-spline basis functions of the space $\mathcal{S}(\mathcal{T}_k^{p-d_\alpha,r}, [0,1])$. For each pair $(g_0, g_1) = (0, \bar{N}_j)$, $j = 0, \dots, \bar{n} - 1$, a basis function $\phi_{1,j}$ is defined via representation (22), i.e.

$$\phi_{1,j}: \; g^{(S)}(u,v) = \frac{\tau_1}{p} \alpha^{(S)}(v) \bar{N}_j(v) N_1^{p,r}(u), \; S \in \{L, R\}.$$

Then the collection of isogeometric functions $\{\phi_{0,i}, \phi_{1,j}\}_{i=0,\dots,\tilde{n}-1; j=0,\dots,\bar{n}-1}$ forms a basis of the space $\mathcal{V}_2^1$.

Remark 13. The basis functions $\phi_{0,i}$ and $\phi_{1,j}$ have small local supports which are comparable with the supports of the basis functions considered in [16, 17], which were constructed for the special case of bilinear geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ with $d_\alpha = 1$, $z_\beta = 0$, $p = 3, 4$, $r = 1$ and $\tau_i = \frac{i}{k+1}$ for $i = 1, \dots, k$.

Remark 14. Our selection of the functions $\beta^{(L)}$ and $\beta^{(R)}$, as described above, is of course only one possibility. It would be even possible to choose for each function $\phi_{0,i}$ a different pair of functions $\beta^{(L)}$ and $\beta^{(R)}$, if desired. In addition, in case of $\beta \neq 0$ and $z_\beta = 1$, the choice of the functions $\beta^{(L)}$ and $\beta^{(R)}$ satisfying $\beta^{(L)}(\tau_\ell) = \beta^{(R)}(\tau_\ell) = 0$, where τ_ℓ with the index $\ell \in \{1, \dots, k\}$ is a root of β, would also lead for this case to the simplified representation (24) for all functions $\phi_{0,i}$.

Example 15. We consider the AS G^1 two-patch geometry $\boldsymbol{F}^{(L)}$, $\boldsymbol{F}^{(R)} \in \mathcal{P}^3([0,1]^2) \times \mathcal{P}^3([0,1]^2)$, which is shown in Fig. 1 and is given by the control points

$$\boldsymbol{c}_{0,0}^{(L)} = (\frac{3}{50}, -\frac{1}{20}), \quad \boldsymbol{c}_{1,0}^{(L)} = -(\frac{774887}{668100}, \frac{19799}{267240}), \quad \boldsymbol{c}_{2,0}^{(L)} = -(\frac{189}{100}, \frac{107}{100}), \quad \boldsymbol{c}_{3,0}^{(L)} = -(\frac{151}{50}, \frac{49}{100}),$$

$$\boldsymbol{c}_{0,1}^{(L)} = (\frac{7}{20}, \frac{24}{25}), \quad \boldsymbol{c}_{1,1}^{(L)} = (-\frac{294947}{334050}, \frac{819}{850}), \quad \boldsymbol{c}_{2,1}^{(L)} = (-\frac{101}{50}, \frac{57}{100}), \quad \boldsymbol{c}_{3,1}^{(L)} = (-\frac{67}{25}, \frac{4}{5}),$$

$$\boldsymbol{c}_{0,2}^{(L)} = (\frac{47}{100}, 2), \quad \boldsymbol{c}_{1,2}^{(L)} = (-\frac{623281}{801720}, \frac{233057}{117900}), \quad \boldsymbol{c}_{2,2}^{(L)} = (-\frac{213}{100}, \frac{109}{50}), \quad \boldsymbol{c}_{3,2}^{(L)} = (-\frac{189}{50}, \frac{203}{100}),$$

$$\boldsymbol{c}_{0,3}^{(L)} = (\frac{1}{20}, \frac{307}{100}), \quad \boldsymbol{c}_{1,3}^{(L)} = (-\frac{422117}{334050}, \frac{1969021}{668100}), \quad \boldsymbol{c}_{2,3}^{(L)} = (-\frac{201}{100}, \frac{189}{50}), \quad \boldsymbol{c}_{3,3}^{(L)} = (-\frac{84}{25}, \frac{331}{100}),$$

and

$$\boldsymbol{c}_{0,0}^{(R)} = \boldsymbol{c}_{0,0}^{(L)}, \quad \boldsymbol{c}_{1,0}^{(R)} = (\frac{787217}{801720}, -\frac{50021}{400860}), \quad \boldsymbol{c}_{2,0}^{(R)} = (\frac{123}{50}, -\frac{61}{100}), \quad \boldsymbol{c}_{3,0}^{(R)} = (\frac{347}{100}, -\frac{6}{25}),$$

$$\boldsymbol{c}_{0,1}^{(R)} = \boldsymbol{c}_{0,1}^{(L)}, \quad \boldsymbol{c}_{1,1}^{(R)} = (\frac{3705053}{3006450}, \frac{2796793}{3006450}), \quad \boldsymbol{c}_{2,1}^{(R)} = (\frac{113}{50}, \frac{17}{20}), \quad \boldsymbol{c}_{3,1}^{(R)} = (\frac{53}{20}, \frac{113}{100}),$$

$$\boldsymbol{c}_{0,2}^{(R)} = \boldsymbol{c}_{0,2}^{(L)}, \quad \boldsymbol{c}_{1,2}^{(R)} = (\frac{581369}{445400}, \frac{24743903}{12025800}), \quad \boldsymbol{c}_{2,2}^{(R)} = (2, \frac{107}{50}), \quad \boldsymbol{c}_{3,2}^{(R)} = (\frac{351}{100}, \frac{9}{4}),$$

$$\boldsymbol{c}_{0,3}^{(R)} = \boldsymbol{c}_{0,3}^{(L)}, \quad \boldsymbol{c}_{1,3}^{(R)} = (\frac{267523}{334050}, \frac{1298303}{400860}, \quad \boldsymbol{c}_{2,3}^{(R)} = (\frac{107}{50}, \frac{319}{100}), \quad \boldsymbol{c}_{3,3}^{(R)} = (\frac{297}{100}, \frac{173}{50}).$$

The corresponding functions $\alpha^{(L)}$, $\alpha^{(R)}$ and β are given by means of Equations (6)-(8) and selecting the function γ as

$$\gamma(v) = \frac{10000}{8167 + 60v - 407v^2 + 516v^3 + 407v^4},$$

which leads to

$$\alpha^{(L)}(v) = -\frac{3}{2}(9+v), \ \alpha^{(R)}(v) = -\frac{3}{2}(-7+v) \text{ and } \beta(v) = \frac{1}{12}(15 - 32v + v^2),$$

respectively. The minimization of (23) leads to

$$\beta^{(L)}(v) = -\frac{83}{1194} + \frac{503v}{3528} \text{ and } \beta^{(R)}(v) = -\frac{23}{597} + \frac{152v}{1791}.$$

Clearly, we have $\boldsymbol{F}^{(L)}, \boldsymbol{F}^{(R)} \in \mathcal{S}(\mathcal{T}_k^{3,1}, [0,1]^2) \times \mathcal{S}(\mathcal{T}_k^{3,1}, [0,1]^2)$ with $k \geq 0$, and we obtain for these selections of the geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ that

$$\dim \mathcal{V}^1 = 2(2+2k)(4+2k) + 7 + 2k = 23 + 26k + 8k^2.$$

Below, let $\mathcal{T}_k^{3,1}$ be the uniform knot vector

$$\mathcal{T}_k^{3,1} = (0,0,0,0, \frac{1}{k+1}, \frac{1}{k+1}, \frac{2}{k+1}, \frac{2}{k+1}, \ldots, \frac{k}{k+1}, \frac{k}{k+1}, 1,1,1,1).$$

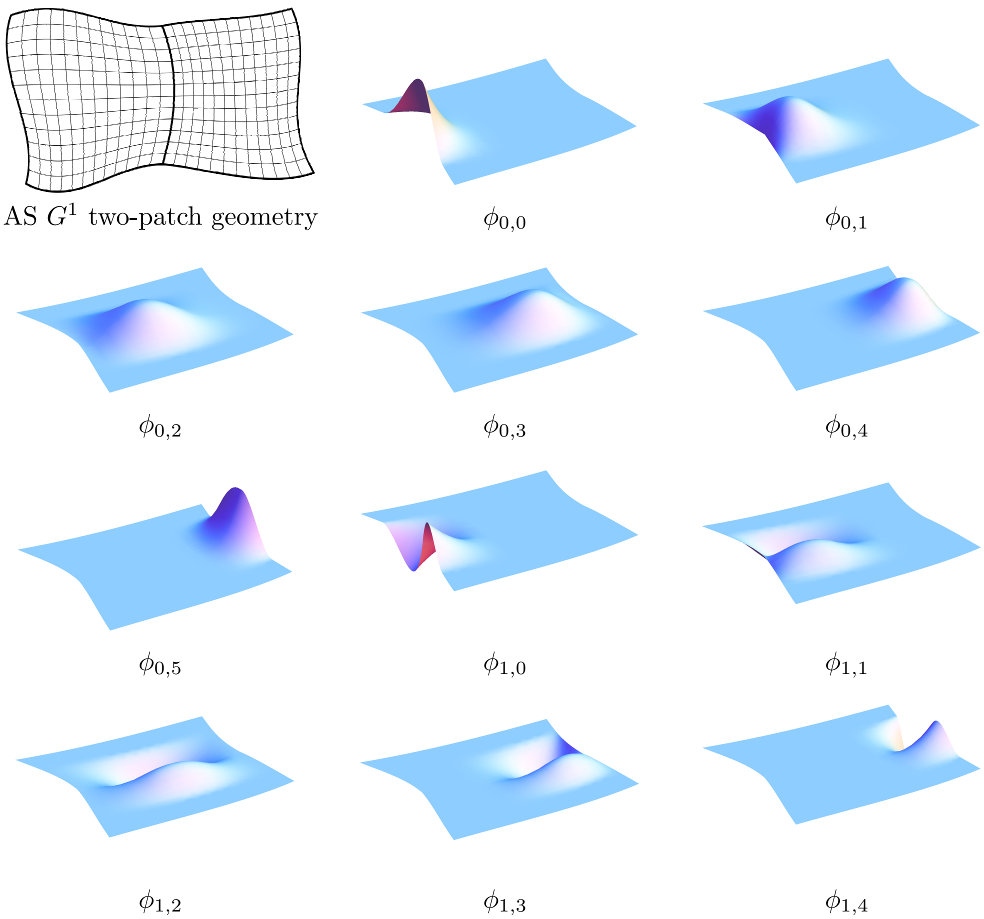

Figure 1: The graphs of the basis functions $\phi_{0,i}$ and $\phi_{1,j}$ of the resulting space $\mathcal{V}_2^1$ for the given bicubic AS G^1 two-patch geometry, when both patches $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ are represented in the space $\mathcal{S}(\mathcal{T}_k^{3,1},[0,1]^2) \times \mathcal{S}(\mathcal{T}_k^{3,1},[0,1]^2)$ for $k=2$. (The graphs are plotted in the parameter range $[0,\frac{1}{2}] \times [0,1]$ for both patches $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$.)

k	0	1	2	3	4	5	10	15	20	30	40
$\mathcal{V}^0$	938.91	724.57	654.29	624.71	608.64	598.63	578.41	572.41	569.93	567.89	567.31
$\mathcal{V}^1$	273.49	425.71	520.62	552.6	564.4	569.07	571.02	569.51	568.51	567.58	567.28

Table 1: For different k, the condition numbers κ of the diagonally scaled mass matrices M for the standard C^0-smooth isogeometric basis functions ($\mathcal{V}^0$) and our C^1-smooth isogeometric basis functions ($\mathcal{V}^1$), cf. Example 15.

Fig. 1 shows the graphs of the resulting basis functions $\phi_{0,i}$ and $\phi_{1,j}$ of the space $\mathcal{V}_2^1$ for $k = 2$.

Let ϕ_i, $i = 0, \ldots, 22 + 26k + 8k^2$, be the C^1-smooth isogeometric basis functions of $\mathcal{V}^1$, which are collected as follows:

$$\phi_i = \begin{cases} \phi^{(L)}_{\lfloor i/(4+2k) \rfloor + 2, i \bmod (4+2k)} & \text{if } 0 \le i \le 7 + 12k + 4k^2 \\ \phi^{(R)}_{\lfloor (i-8-12k-4k^2)/(4+2k) \rfloor + 2, i \bmod (4+2k)} & \text{if } 8 + 12k + 4k^2 \le i \le 15 + 24k + 8k^2 \\ \phi_{0,i-16-24k-8k^2} & \text{if } 16 + 24k + 8k^2 \le i \le 19 + 25k + 8k^2, \\ \phi_{1,i-20-25k-8k^2} & \text{if } 20 + 25k + 8k^2 \le i \le 22 + 26k + 8k^2. \end{cases}$$

In addition, we denote by $g_i^{(S)}$, $S \in \{L, R\}$, the associated spline functions $\phi_i \circ \boldsymbol{F}^{(S)}$. Let us consider the mass matrix $M = (m_{i,j})_{i,j \in \{0,\ldots,22+26k+8k^2\}}$ with the entries

$$m_{i,j} = \int_\Omega \phi_i(\boldsymbol{x})\phi_j(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{S \in \{L,R\}} \int_0^1 \int_0^1 g_i^{(S)}(u,v) g_j^{(S)}(u,v) |\det(J^{(S)}(u,v))| \mathrm{d}u\mathrm{d}v,$$

where $J^{(S)}$, $S \in \{L, R\}$, is the Jacobian of $\boldsymbol{F}^{(S)}$. We also compute (for comparison) the mass matrix for the standard C^0-smooth isogeometric basis functions of the space $\mathcal{V}^0 = \mathcal{V} \cap C^0(\Omega)$. Table 1 reports for different k the condition numbers κ of the diagonally scaled mass matrices M (cf. [7]) for the two different bases. The results indicate that the basis functions ϕ_i are as well conditioned as the standard C^0-smooth isogeometric basis functions.

6. Spline coefficients of the basis functions of the space $\mathcal{V}_2^1$

We represent the spline functions $g^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, for the previous constructed isogeometric basis functions ϕ of the space $\mathcal{V}_2^1$ as a linear combination of the tensor-product B-splines $N_{i,j}^{p,r}$. Thereby, these linear factors (i.e. the B-spline coefficients of the spline functions $g^{(S)}$ with respect to the space $\mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2)$) will be described by means of blossoming. For the sake of simplicity (especially with respect to notation), we will restrict ourselves below to the case $\beta = 0$ or $z_\beta = 0$. Note that our framework could be also extended to the remaining cases.

6.1. Concept of blossoming

We give a short overview of the concept of blossoming. For more detail we refer to e.g. [8, 11, 21, 29, 30]. Given a univariate spline function $h \in \mathcal{S}(\mathcal{T}_k^{p,r})$ with the spline representation (1), there exists a uniquely defined function $H : \mathbb{R}^p \to \mathbb{R}$, called the blossom of h, possessing the following properties:

- H is symmetric,
- H is multi-affine, and
- $H(t, \dots, t) = h(t)$.

These properties imply (by the so called dual function property, see [11])

$$H(t_{i+1}^{p,r}, \dots, t_{i+p}^{p,r}) = d_i,\ \ i = 0, \dots, p + k(p - r),$$

and therefore fully determine the blossom, since the value $H(s_1, \dots, s_p)$ for arbitrary values $s_1, \dots, s_p$ can be computed by recursively using the convex combinations

$$H(t_{i+1}^{p,r}, \dots, t_{i+p-m}^{p,r}, s_1, \dots, s_m) =$$
$$(1 - \gamma_i^m(s_m))H(t_i^{p,r}, \dots, t_{i+p-m}^{p,r}, s_1, \dots, s_{m-1}) + \gamma_i^m(s_m)H(t_{i+1}^{p,r}, \dots, t_{i+p-m+1}^{p,r}, s_1, \dots, s_{m-1}),$$

for $i = m, \dots, p + kp - r$, $m = 1, \dots, p$ and

$$\gamma_i^m(s) = \frac{s - t_i^{p,r}}{t_{i+p-m+1}^{p,r} - t_i^{p,r}}.$$

The concept of blossoming provides a simple way to perform knot insertion, to differentiate a spline function and to multiply two spline functions. Given the spline function $h \in \mathcal{S}(\mathcal{T}_k^{p,r}, [0, 1])$ with the blossom H. Representing h as a spline function in the space $\mathcal{S}(\mathcal{T}_k^{p,r-1}, [0, 1])$, the corresponding spline control points $\bar{d}_i$ are given by

$$\bar{d}_i = H(t_{i+1}^{p,r-1}, \dots, t_{i+p}^{p,r-1}), \quad i = 0, \dots, p + k(p - r + 1).$$

The spline control points $\tilde{d}_i$ of the derivative of h, i.e. $h' \in \mathcal{S}(\mathcal{T}_k^{p-1,r-1}, [0, 1])$, can be computed as follows:

$$\tilde{d}_i = \frac{p}{t_{i+p+1}^{p,r} - t_{i+1}^{p,r}}(H(t_{i+1}^{p-1,r-1}, \dots, t_{i+p-1}^{p-1,r-1}, t_{i+p+1}^{p,r}) - H(t_{i+1}^{p-1,r-1}, \dots, t_{i+p-1}^{p-1,r-1}, t_{i+1}^{p,r}))$$

for $i = 0, \dots, p - 1 + k(p - r)$. Given further the spline function $h_1 \in \mathcal{S}(\mathcal{T}_k^{p_1,r_1}, [0, 1])$ with the blossom H_1. Let $\hat{p} = p + p_1$ and $\hat{r} = \min(r, r_1)$. Then the spline control points $\hat{d}_i$ of the product $\hat{h} = hh_1 \in \mathcal{S}(\mathcal{T}_k^{\hat{p},\hat{r}}, [0, 1])$ are given by

$$\hat{d}_i = \frac{1}{\binom{\hat{p}}{p}} \sum H(t_{i_1}^{\hat{p},\hat{r}}, \dots, t_{i_p}^{\hat{p},\hat{r}})H_1(t_{i_{p+1}}^{\hat{p},\hat{r}}, \dots, t_{i_{\hat{p}}}^{\hat{p},\hat{r}}), \quad i = 0, \dots, \hat{p} + k(\hat{p} - \hat{r}),$$

where the summation runs over all possibilities to split the set $\{i + 1, \dots, i + \hat{p}\}$ into the two disjoint subsets $\{i_1, \dots, i_p\}$ and $\{i_{p+1}, \dots, i_{\hat{p}}\}$.

6.2. Spline coefficients as matrix entries

Let $n = p+1+k(p-r)$, $\tilde{n} = p+1+k(p-r-1)+z_\beta$ and $\bar{n} = p+1-d_\alpha+k(p+1-d_\alpha-r)$. We denote by $g_{0,i}^{(S)}$, $i = 0,\ldots,\tilde{n}-1$, and $g_{1,j}^{(S)}$, $j = 0,\ldots,\bar{n}-1$, for $S \in \{L,R\}$ the spline functions $\phi_{0,i} \circ \boldsymbol{F}^{(S)}$, and $\phi_{1,j} \circ \boldsymbol{F}^{(S)}$, respectively. In addition, let $\boldsymbol{B}^{(S)} = (\boldsymbol{B}_0^{(S)}, \boldsymbol{B}_1^{(S)})^T$ be the vector of functions, where

$$\boldsymbol{B}_0^{(S)}(u,v) = (g_{0,0}^{(S)}(u,v), \ldots, g_{0,\tilde{n}-1}^{(S)}(u,v))^T,$$

as well as

$$\boldsymbol{B}_1^{(S)}(u,v) = (g_{1,0}^{(S)}(u,v), \ldots, g_{1,\bar{n}-1}^{(S)}(u,v))^T,$$

and let $\boldsymbol{B}^* = (\boldsymbol{B}_0^*, \boldsymbol{B}_1^*)^T$ be the vector of functions, where

$$\boldsymbol{B}_0^*(u,v) = (N_{0,0}^{p,r}(u,v), \ldots, N_{0,n-1}^{p,r}(u,v))^T,$$

as well as

$$\boldsymbol{B}_1^*(u,v) = (N_{1,0}^{p,r}(u,v), \ldots, N_{1,n-1}^{p,r}(u,v))^T.$$

Then there exists a matrix $A^{(S)} \in \mathbb{R}^{(\tilde{n}+\bar{n})\times 2n}$ such that

$$\boldsymbol{B}^{(S)}(u,v) = A^{(S)}\boldsymbol{B}^*(u,v).$$

The matrix $A^{(S)}$ for $S \in \{L,R\}$ has a block structure, resulting in an equation of the form

$$\begin{pmatrix} \boldsymbol{B}_0^{(S)} \\ \boldsymbol{B}_1^{(S)} \end{pmatrix} = \begin{pmatrix} A_1 & A_2^{(S)} \\ 0 & A_3^{(S)} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_0^* \\ \boldsymbol{B}_1^* \end{pmatrix}, \tag{25}$$

where $A_1 \in \mathbb{R}^{\tilde{n}\times n}$, $A_2^{(S)} \in \mathbb{R}^{\tilde{n}\times n}$ and $A_3^{(S)} \in \mathbb{R}^{\bar{n}\times n}$. For each row the single entries of the matrix $A^{(S)}$, $S \in \{L,R\}$, provide the B-spline coefficients of the corresponding spline function $g_{0,i}^{(S)}$ or $g_{1,j}^{(S)}$ with respect to the spline space $\mathcal{S}(\mathcal{T}_k^{p,r}, [0,1]^2)$.

The following lemma provides the entries of the single matrices A_1, $A_2^{(S)}$ and $A_3^{(S)}$:

Lemma 16. *We denote by $H_i^{p,r}$ the blossom of the B-spline $N_i^{p,r}$. For a linear function $w : [0,1] \to \mathbb{R}$ with the Bézier representation*

$$w(t) = w_0(1-t) + w_1 t, \quad t \in [0,1], \quad w_0, w_1 \in \mathbb{R}, \tag{26}$$

we define the matrix $\hat{A}(w) = (\hat{a}_{i,j}^{(w)})_{i,j}$ given by

$$\hat{a}_{i,j}^{(w)} = \frac{1}{p}\sum_{l=1}^{p}((1-t_{j+l}^{p,r})w_0 + t_{j+l}^{p,r}w_1)H_i^{p-1,r}(t_{j+1}^{p,r}, \ldots, t_{j+l-1}^{p,r}, t_{j+l+1}^{p,r}, \ldots, t_{j+p}^{p,r}).$$

The matrices A_1, $A_2^{(S)}$ and $A_3^{(S)}$, compare (25), are given as follows (depending on $\alpha^{(S)}$, $\beta^{(S)}$ or β):

- *Let $\beta = 0$. Then we have*

$$A_1 = A_2^{(S)} = I_n, \text{ and } A_3^{(S)} = \begin{cases} \frac{\tau_1}{p}\alpha^{(S)} I_n & \text{if } d_\alpha = 0, \\ \frac{\tau_1}{p}\hat{A}(\alpha^{(S)}) & \text{if } d_\alpha = 1. \end{cases}$$

- *Let $z_\beta = 0$. Then we have*

$$A_1 = \bar{A}, \; A_2^{(S)} = \bar{A} + \frac{\tau_1}{p}\tilde{A}\hat{A}(\beta^{(S)}), \text{ and } A_3^{(S)} = \begin{cases} \frac{\tau_1}{p}\alpha^{(S)} I_n & \text{if } d_\alpha = 0, \\ \frac{\tau_1}{p}\hat{A}(\alpha^{(S)}) & \text{if } d_\alpha = 1, \end{cases}$$

  where the entries of the matrices $\bar{A} = (\bar{a}_{i,j})_{i,j}$ and $\tilde{A} = (\tilde{a}_{i,j})_{i,j}$ are given by

$$\bar{a}_{i,j} = H_i^{p,r+1}(t_{j+1}^{p,r}, \dots, t_{j+p}^{p,r})$$

  and

$$\tilde{a}_{i,j} = \frac{p}{t_{j+p+1}^{p,r+1} - t_{j+1}^{p,r+1}}(H_i^{p,r+1}(t_{j+1}^{p-1,r}, \dots, t_{j+p-1}^{p-1,r}, t_{j+p+1}^{p,r+1}) - H_i^{p,r+1}(t_{j+1}^{p-1,r}, \dots, t_{j+p-1}^{p-1,r}, t_{j+1}^{p,r+1})),$$

  respectively.

Here, I_n is the identity matrix of dimension n.

Proof. The results follow directly from the concept of blossoming as presented in Subsection 6.1. □

Remark 17. The matrices A_1, $A_2^{(S)}$ and $A_3^{(S)}$, see (25), are sparse matrices (compare Example 19).

Remark 18. A further possibility to construct the matrices A_1, $A_2^{(S)}$ and $A_3^{(S)}$, see (25), is the use of the concept of fitting. Thereby, the m-th row of the matrices A_1, $A_2^{(S)}$ and $A_3^{(S)}$, denoted by $(a_{1,m,0}, \dots, a_{1,m,n-1})$, $(a_{2,m,0}^{(S)}, \dots, a_{2,m,n-1}^{(S)})$ and $(a_{3,m,0}^{(S)}, \dots, a_{3,m,n-1}^{(S)})$, respectively, are computed by minimizing the terms

$$\sum_{i=0}^{n-1}(g_{0,m}^{(L)}(0,\xi_i) - \sum_{j=0}^{n-1} a_{1,m,j} N_j^{p,r}(\xi_i))^2,$$

$$\sum_{i=0}^{n-1}(\tau_1 \frac{D_u g_{0,m}^{(S)}(0,\xi_i)}{p} + g_{0,m}^{(S)}(0,\xi_i) - \sum_{j}^{n-1} a_{2,m,j}^{(S)} N_j^{p,r}(\xi_i))^2$$

and

$$\sum_{i=0}^{n-1}(\tau_1 \frac{D_u g_{1,m}^{(S)}(0,\xi_i)}{p} + g_{1,m}^{(S)}(0,\xi_i) - \sum_{j=0}^{n-1} a_{3,m,j}^{(S)} N_j^{p,r}(\xi_i))^2,$$

respectively, where $(\xi_i)_{i=0,\dots,n-1}$ are the Greville abscissa of the B-splines $N_j^{p,r}$, $j = 0, \dots, n-1$, of the spline space $\mathcal{S}(\mathcal{T}_k^{p,r}, [0,1])$.

Example 19. Let $\beta \neq 0$ and $z_\beta = 0$, $p = 3$, $r = 1$, $k \geq 2$ and $\tau_i = \frac{i}{k+1}$ for $i = 1, \dots, k$. Let $w : [0,1] \to \mathbb{R}$ be a linear function with the Bézier representation (26). Then the rows $\bar{a}_i$, $\tilde{a}_i$ and $\hat{a}_i^{(w)}$ of the matrices $\bar{A}$, $\tilde{A}$ and $\hat{A}(w)$ from Lemma 16 are given as follows:

$$\bar{a}_i = \begin{cases} (1, \underbrace{0, \dots, 0}_{2k+3}) & i = 0 \\ (0, 1, \frac{1}{2}, \underbrace{0, \dots, 0}_{2k+1}) & i = 1 \\ (0, 0, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \underbrace{0, \dots, 0}_{2k-1}) & i = 2 \\ (\underbrace{0, \dots, 0}_{2i-3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \underbrace{0, \dots, 0}_{2(k-i)+3}) & 3 \leq i \leq k \\ (\underbrace{0, \dots, 0}_{2k-1}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, 0, 0) & i = k+1 \\ (\underbrace{0, \dots, 0}_{2k+1}, \frac{1}{2}, 1, 0) & i = k+2 \\ (\underbrace{0, \dots, 0}_{2k+3}, 1) & i = k+3 \end{cases}, \quad \tilde{a}_i = \begin{cases} (-3(k+1), \underbrace{0, \dots, 0}_{k+2}) & i = 0 \\ (3(k+1), -\frac{3(k+1)}{2}, \underbrace{0, \dots, 0}_{k+1}) & i = 1 \\ (0, \frac{3(k+1)}{2}, -(k+1), \underbrace{0, \dots, 0}_{k}) & i = 2 \\ (\underbrace{0, \dots, 0}_{i-1}, k+1, -(k+1), \underbrace{0, \dots, 0}_{k-i+2}) & 3 \leq i \leq k \\ (\underbrace{0, \dots, 0}_{k}, k+1, -\frac{3(k+1)}{2}, 0) & i = k+1 \\ (\underbrace{0, \dots, 0}_{k+1}, \frac{3(k+1)}{2}, -3(k+1)) & i = k+2 \\ (\underbrace{0, \dots, 0}_{k+2}, 3(k+1)) & i = k+3 \end{cases}$$

and

$$\hat{a}_i^{(w)} = \frac{1}{6(k+1)} * \begin{cases} (6(k+1)w_0, 2kw_0 + 2w_1, \underbrace{0, \dots, 0}_{2k+2}) & i = 0 \\ (0, 4(k+1)w_0, (1+5k)w_0 + 4w_1, (k-1)w_0 + 2w_1, \underbrace{0, \dots, 0}_{2k}) & i = 1 \\ \left\{ \begin{array}{l} (\underbrace{0, \dots, 0}_{2(i-1)}, (k+3-i)w_0 + (i-2)w_1, (9+5(k-i))w_0 + (6+5(i-2))w_1, \\ (6+5(k-i))w_0 + (9+5(i-2))w_1, (k-i)w_0 + (i+1)w_1, \underbrace{0, \dots, 0}_{2(k-i+1)}) \end{array} \right\} & 2 \leq i \leq k \\ (\underbrace{0, \dots, 0}_{2k}, 2w_0 + (k-1)w_1, 4w_0 + (1+5k)w_1, 4(k+1)w_1, 0) & i = k+1 \\ (\underbrace{0, \dots, 0}_{2k+2}, 2w_0 + 2kw_1, 6(k+1)w_1) & i = k+2 \end{cases}$$

7. Conclusion

We have studied the spaces of C^1-smooth isogeometric functions over a special class of two-patch geometries, so-called analysis-suitable G^1 (AS G^1) two-patch parameterizations (cf. [9]). This class of two-patch geometries is of particular interest, since exactly these geometries allow under certain assumptions C^1 isogeometric spaces with optimal approximation properties, see [9]. More precisely, we have computed the dimension of these C^1 spaces and have presented an explicit basis construction. The resulting basis functions are well conditioned, have small local supports and their spline coefficients can be simply computed by means of blossoming or fitting.

Note that the constructed basis interpolates traces and transversal derivatives at the interface. Hence, the basis functions may be negative. In fact, the functions interpolating the transversal derivative are by construction positive on one side of the interface and negative on the other. The presented basis can be transformed easily to obtain locally supported basis functions which sum up to one. However, it is unclear whether or not a non-negative, local partition of unity exists for all AS G^1 parameterizations. This will be of interest for future research.

One issue that remains to be studied is the flexibility of AS G^1 geometries over general multi-patch domains. The basis construction over two-patch geometries can be applied also to multi-patch configurations except for the basis functions around vertices, where modifications might be necessary. We are confident that the presented basis representation can be extended to the multi-patch case and used for fitting procedures, that approximate any given geometry with an AS G^1 parameterization.

The developed basis provides a simple representation that can be implemented in existing IGA libraries. Thus, our C^1-smooth functions may be used to discretize different fourth-order partial differential equations. It may be of interest for future studies to perform such simulations and analyze their properties as well as to investigate the class of AS G^1 parameterizations for volumetric two-patch and multi-patch domains.

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