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Orthogonality of the Dickson Polynomials of the $(k+1)$-th Kind

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Abstract

We study the Dickson polynomials of the $(k+1)$-th kind over the field of complex numbers. We show that they are a family of orthogonal polynomials with respect to a quasi-definite moment functional L. We find an integral representation for L and compute explicit expressions for all of its moments.

Dedicated to V. E. P., in gratitude for her continuous unbounded support (without measure).

Keywords: Dickson polynomials, orthogonal polynomials, moment functional, Stieltjes transform.

Classification codes: 11T06 (primary), 33C45, 26A42, 28C05 (secondary).

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1 Introduction

Let $n \in \mathbb{N}$, $\mathbb{F}_q$ be a finite field and $a \in \mathbb{F}_q$. The Dickson polynomials $D_n(x;a)$, defined by [15, 9.6.1]

$$D_n(x;a) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} (-a)^j \, x^{n-2j}, \quad x \in \mathbb{F}_q$$

where introduced by Leonard Eugene Dickson (1874 – 1954) in his 1896 Ph.D. thesis "The analytic representation of substitutions on a power of a prime number of letters, with a discussion of the linear group" [5], published in two parts in The Annals of Mathematics [6], [7]. The Dickson polynomials are the unique monic polynomials satisfying the functional equation [15, 9.6.3]

$$D_n\left(y + \frac{a}{y}; a\right) = y^n + \left(\frac{a}{y}\right)^n, \quad y \in \mathbb{F}_{q^2}.$$

See [14] for further algebraic and number theoretic properties of the Dickson polynomials.

In [21], Wang and Yucas extended the Dickson polynomials to a family depending on a new parameter $k = 0, 1, \ldots,$ which they called *Dickson polynomials of the* $(k+1)$*-th kind.* They defined them by

$$D_{n,k}(x;a) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-kj}{n-j} \binom{n-j}{j} (-a)^j \, x^{n-2j}, \tag{1}$$

with initial values

$$D_{0,k}(x;a) = 2-k, \quad D_{1,k}(x;a) = x. \tag{2}$$

They also showed that the polynomials $D_{n,k}(x;a)$ satisfy the fundamental functional equation

$$D_{n,k}(y + \frac{a}{y}; a) = y^n + \left(\frac{a}{y}\right)^n + k \frac{ay^n - y^2 \left(\frac{a}{y}\right)^n}{y^2 - a}, \quad y \neq 0. \tag{3}$$

Note that

$$\lim_{y \to \pm\sqrt{a}} \frac{ay^n - y^2 \left(\frac{a}{y}\right)^n}{y^2 - a} = (n-1) \left(\pm\sqrt{a}\right)^n,$$

and therefore

$$D_{n,k}(y+\frac{a}{y};a)=[2+(n-1)\,k]\left(\pm\sqrt{a}\right)^n,\quad y=\pm\sqrt{a}.$$

We clearly have

$$D_{n,0}(x;a)=D_n(x;a)\qquad\text{(Dickson polynomials)}$$

and

$$D_{n,1}(x;a)=E_n(x;a)\qquad\text{(Dickson polynomials of the second kind).}$$

In fact, since

$$\frac{n-kj}{n-j}=k-(k-1)\,\frac{n}{n-j},$$

we have [21, 2.1]

$$D_{n,k}(x;a)=kE_n(x;a)-(k-1)\,D_n(x;a).$$

The polynomials $D_{n,k}(x;a)$ also satisfy the recurrence (see [21, Remark 2.5])

$$D_{n+2,k}=xD_{n+1,k}-aD_{n,k}. \tag{4}$$

The first few Dickson polynomials of the $(k+1)$-th kind are

$$\begin{aligned}
D_{2,k}(x;a)&=x^2+a\,(k-2)\,,\\
D_{3,k}(x;a)&=x^3+a\,(k-3)\,x,\\
D_{4,k}(x;a)&=x^4+a\,(k-4)\,x^2+a^2\,(2-k)\,,\\
D_{4,k}(x;a)&=x^5+a\,(k-5)\,x^3+a^2\,(5-2k)\,x.
\end{aligned}$$

They have zeros at

$$\begin{aligned}
&x=\pm\sqrt{a}\sqrt{2-k},\quad \text{if } n=2,\\
&x=0,\quad \pm\sqrt{a}\sqrt{3-k},\quad \text{if } n=3,\\
&x=\pm\frac{\sqrt{a}}{\sqrt{2}}\sqrt{4-k\pm\sqrt{(k-2)^2+4}},\quad \text{if } n=4,\\
&x=0,\quad \pm\frac{\sqrt{a}}{\sqrt{2}}\sqrt{5-k\pm\sqrt{(k-1)^2+4}},\quad \text{if } n=5,
\end{aligned} \tag{5}$$

as can be verified using a mathematical symbolic computation program such as Mathematica.

In this article, we study the polynomials $D_{n,k}(x;a)$ over the field of complex numbers, with $a > 0$ and $k \in \mathbb{R}$. Our motivation is the three-term recurrence relation (4), which suggests that the Dickson polynomials of the $(k+1)$-th kind form a family of orthogonal polynomials with respect to some linear functional L. However, from (5) we see that for $k > 2$ the polynomials $D_{n,k}(x;a)$ may have a pair of purely imaginary roots. Also, the polynomials $D_{3,3}(x;a)$, $D_{4,2}(x;a)$, and $D_{5,\frac{5}{2}}(x;a)$ have a triple zero at $x = 0$. This implies that the linear functional L is quasi-definite [4, Theorem 2.4.3], [9, Theorem 1].

The article is organized as follows: in Section 2, we derive some of the main properties of the Dickson polynomials of the $(k+1)$-th kind, including different expressions, a hypergeometric representation, differential equations, and a generating function.

In Section 3, we present some basic results from the theory of orthogonal polynomials that we will need to find the linear functional L. We also stress the connection between the polynomials $D_{n,k}(x;a)$ and the Chebyshev polynomials of the first and second kind.

In Section 4 we introduce a family of orthogonal polynomials related to $D_{n,k}(x;a)$ and independent of a. They greatly simplify some of the computations needed to find L. We apply our results to the Dickson polynomials of the $(k+1)$-th kind and obtain a representation for their moment functional L. We also find explicit expressions for the moments of L.

Finally, in Section 5 we summarize our results. In our hope that the results would be of interest to researchers outside the field of orthogonal polynomials and special functions, we have made the paper as self-contained as possible.

2 Properties

We begin by checking the initial polynomial $D_{0,k}(x;a)$. It is not clear from the definition (1) that $D_{0,k}(x;a) = 2 - k$, but if we consider even and odd degrees, we have the following result.

Proposition 1 *The even and odd Dickson polynomials of the* $(k+1)$*-th kind* *are given by*

$$D_{2n,k}(x;a) = (2-k)\,(-a)^n + \sum_{l=1}^{n} \frac{(2-k)\,n + kl}{l+n} \binom{n+l}{2l} (-a)^{n-l}\, x^{2l} \tag{6}$$

and

$$D_{2n+1,k}(x;a) = x \sum_{l=0}^{n} \frac{(2-k)\,n + kl + 1}{l+n+1} \binom{n+l+1}{2l+1} (-a)^{n-l}\, x^{2l}. \tag{7}$$

Proof. From (1), we have

$$D_{2n,k}(x;a) = \sum_{j=0}^{n} \frac{2n-kj}{2n-j} \binom{2n-j}{j} (-a)^j\, x^{2n-2j},$$

and switching the index to $l = n - j$, we get

$$D_{2n,k}(x;a) = \sum_{l=0}^{n} \frac{(2-k)\,n + kl}{l+n} \binom{n+l}{n-l} (-a)^{n-l}\, x^{2l}.$$

Finally, we use the symmetry of the binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}.$$

A similar calculation gives (7). ■

Next, we will find a representation for $D_{n,k}(x;a)$ in terms of the generalized hypergeometric function

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; x \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{x^j}{j!},$$

where $(u)_j$ denotes the Pochhammer symbol (also called shifted or rising factorial) defined by [16, 5.2.4]

$$\begin{aligned} (a)_0 &= 1 \\ (a)_j &= a\,(a+1) \cdots (a+j-1)\,, \quad j \in \mathbb{N}. \end{aligned}$$

Proposition 2 *The Dickson polynomials of the $(k+1)$-th kind admit the hypergeometric representation*

$$D_{n,k}(x;a) = x^n \ {}_3F_2\left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2}, 1-\frac{n}{k} \\ 1-n, -\frac{n}{k} \end{matrix} ; \frac{4a}{x^2} \right), \quad k \neq 0. \tag{8}$$

We also have

$$D_{n,0}(x;a) = x^n \ {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2} \\ 1-n \end{matrix} ; \frac{4a}{x^2} \right).$$

Proof. Let

$$D_{n,k}(x;a) = x^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_j,$$

with

$$c_j = \frac{n-kj}{n-j} \binom{n-j}{j} (-a)^j \, x^{-2j}. \tag{9}$$

We have $c_0 = 1$ and $c_j = 0$ for $j > \frac{n}{2}$. Using (9), we get

$$\frac{c_{j+1}}{c_j} = \frac{(2j-n+1)(2j-n)(jk+k-n)}{(j-n+1)(jk-n)(j+1)} \frac{a}{x^2}. \tag{10}$$

Let $k \neq 0$. Then,

$$\frac{c_{j+1}}{c_j} = \frac{\left(j-\frac{n}{2}+\frac{1}{2}\right)\left(j-\frac{n}{2}\right)\left(j+1-\frac{n}{k}\right)}{(j-n+1)\left(j-\frac{n}{k}\right)(j+1)} \frac{4a}{x^2},$$

and we obtain

$$c_j = \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j \left(1-\frac{n}{k}\right)_j}{(1-n)_j \left(-\frac{n}{k}\right)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j.$$

Thus,

$$D_{n,k}(x;a) = x^n \sum_{j=0}^{\infty} \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j \left(1-\frac{n}{k}\right)_j}{(1-n)_j \left(-\frac{n}{k}\right)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j.$$

If $k = 0$, we see from (10) that

$$\frac{c_{j+1}}{c_j} = \frac{4\left(j+\frac{1-n}{2}\right)\left(j-\frac{n}{2}\right)}{(j-n+1)(j+1)} \frac{a}{x^2},$$

and therefore

$$c_j = \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j}{(1-n)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j .$$

Hence,

$$D_{n,0}(x;a) = x^n \sum_{j=0}^{\infty} \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j}{(1-n)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j .$$

■

Remark 3 *Note that from (8) it follows that*

$$D_{2n,k}(0;a) = (2-k)(-a)^n ,$$

in agreement with (6).

Proposition 4 *For $n = 1, 2, \ldots,$ the Dickson polynomials of the $(k+1)$-th kind satisfy the following relations:*

$$D_{n,k+2} - 2D_{n,k+1} + D_{n,k} = 0,$$

$$\begin{aligned} &-(x^2-4a)\left[(k-1)nx^2 + a(k-2)(kn-2n-k)\right] D''_{n,k} \\ &+x\left[(k-1)nx^2 + a\left(6k+4n+3k^2n-4kn-3k^2\right)\right] D'_{n,k} \\ &+\left[(k-1)n^3x^2 + an\left(-k-2n+kn\right)\left(-2k-2n+kn\right)\right] D_{n,k} = 0, \end{aligned} \tag{11}$$

$$\begin{gathered} (x^2-4a)^2 D^{(iv)}_{n,k} + 10x(x^2-4a)D'''_{n,k} + \left[(23-2n^2)x^2 + 8a(n^2-4)\right] D''_{n,k} \\ -3(2n^2-3)xD'_{n,k} + n^2(n^2-4)D_{n,k} = 0, \end{gathered}$$

and

$$(x^2-4a)D''_{n,k} - 4nD_{n+1,k}D'_{n,k} + (2n+3)\,xD'_{n,k} + n\,(n+2)\,D_{n,k} = 0.$$

Proof. All the identities can be automatically found and proved using the hypergeometric representation (8) and the Mathematica package HolonomicFunctions [12]. ■

Remark 5 *The differential equation (11) already appeared in [21, Lemma 2.7].*

We can use the recurrence relation (4) to obtain a different representation for the polynomials $D_{n,k}(x;a)$.

Proposition 6 *For $x \neq \pm 2\sqrt{a}$, the Dickson polynomials of the $(k+1)$-th kind are given by*

$$D_{n,k}(x;a) = \left(1 + k\frac{x-\Delta}{2\Delta}\right)\left(\frac{x+\Delta}{2}\right)^n + \left(1 - k\frac{x+\Delta}{2\Delta}\right)\left(\frac{x-\Delta}{2}\right)^n, \quad (12)$$

where

$$\Delta = \sqrt{x^2 - 4a}.$$

We also have,

$$D_{n,k}(\pm 2\sqrt{a}; a) = (kn + 2 - k)\left(\pm\sqrt{a}\right)^n. \quad (13)$$

Proof. Let us assume that we can write

$$D_{n,k}(x;a) = R^n, \quad (14)$$

for some function $R(x,k,a)$. Using (14) in the recurrence (4), we obtain

$$R^2 - xR + a = 0,$$

and therefore

$$R_{\pm} = \frac{x \pm \Delta}{2}$$

with

$$\Delta = \sqrt{x^2 - 4a}.$$

It follows that the general solution of (4) is given by

$$D_{n,k}(x;a) = C_1(x;a,k)\left(\frac{x+\Delta}{2}\right)^n + C_2(x;a,k)\left(\frac{x-\Delta}{2}\right)^n. \quad (15)$$

Using the initial conditions (2) in (15), we get

$$C_1(x;a,k) + C_2(x;a,k) = 2 - k,$$
$$C_1(x;a,k)\left(\frac{x+\Delta}{2}\right) + C_2(x;a,k)\left(\frac{x-\Delta}{2}\right) = x.$$

Thus, assuming that $x \neq \pm 2\sqrt{a}$,

$$C_1(x;a,k) = 1 + k\frac{x-\Delta}{2\Delta}, \quad C_2(x;a,k) = 1 - k\frac{x+\Delta}{2\Delta}.$$

To verify (13), we replace it in the recurrence (4), and obtain

$$(kn+2+k)\left(\pm\sqrt{a}\right)^{n+2} - \left(\pm 2\sqrt{a}\right)(kn+2)\left(\pm\sqrt{a}\right)^{n+1} + a(kn+2-k)\left(\pm\sqrt{a}\right)^{n}$$
$$= a\left(\pm\sqrt{a}\right)^{n}\left[(kn+2+k) - 2(kn+2) + (kn+2-k)\right] = 0.$$

■

Using (12), we can obtain a generating function for the polynomials $D_{n,k}(x;a)$.

Proposition 7 *The ordinary generating function of the polynomials $D_{n,k}(x;a)$ is given by*

$$G(z;x,k,a) = \sum_{n=0}^{\infty} D_{n,k}(x;a) z^n = \frac{2-k+(k-1)xz}{az^2 - xz + 1}. \tag{16}$$

Proof. From (12), we have (as formal power series)

$$\begin{aligned} G(z;x,k,a) &= \sum_{n=0}^{\infty} D_{n,k}(x;a) z^n \\ &= \left(1 + k\frac{x-\Delta}{2\Delta}\right)\sum_{n=0}^{\infty}\left(z\frac{x+\Delta}{2}\right)^n + \left(1 - k\frac{x+\Delta}{2\Delta}\right)\sum_{n=0}^{\infty}\left(z\frac{x-\Delta}{2}\right)^n \\ &= \left(1 + k\frac{x-\Delta}{2\Delta}\right)\frac{1}{1-\left(z\frac{x+\Delta}{2}\right)} + \left(1 - k\frac{x+\Delta}{2\Delta}\right)\frac{1}{1-\left(z\frac{x-\Delta}{2}\right)}. \end{aligned}$$

Thus,

$$G(z;x,k,a) = 4\frac{2-k+(k-1)xz}{(x^2-\Delta^2)z^2 - 4xz + 4}$$

and the result follows since

$$\Delta^2 = x^2 - 4a.$$

■

Remark 8 *The same generating function was obtained in [21, Lemma 2.6] using the recurrence (4).*

3 Orthogonal polynomials

Let $\{\mu_n\}$ be a sequence of complex numbers and $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ be a linear functional defined by

$$\mathcal{L}[x^n] = \mu_n, \quad n = 0, 1, \ldots.$$

Then, $\mathcal{L}$ is called the *moment functional* determined by the formal moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n.

A moment functional $\mathcal{L}$ is called *positive-definite* if $\mathcal{L}[q(x)] > 0$ for every polynomial $q(x)$ that is not identically zero and is non-negative for all real x. Otherwise, $\mathcal{L}$ is called *quasi-definite*.

A sequence $\{P_n\} \subset \mathbb{C}[x]$, with $\deg(P_n) = n$ is called an *orthogonal polynomial sequence* with respect to $\mathcal{L}$ provided that [3]

$$\mathcal{L}[P_n P_m] = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots,$$

where $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta.

One of the fundamental properties of orthogonal polynomials is that they satisfy a three-term recurrence relation.

Theorem 9 *Let $\mathcal{L}$ be a moment functional and let $\{P_n\}$ be the sequence of monic orthogonal polynomials associated with it. Then, there exist $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C} \setminus \{0\}$ such that*

$$P_{n+1} = (x - \beta_n) P_n - \gamma_n P_{n-1}, \quad n = 1, 2, \ldots. \tag{17}$$

Proof. See [3, Theorem 4.1]. ■

Remark 10 *In this paper, we will say that $\{P_n\}$ is a sequence of "monic polynomials" if* $\deg(P_n) = n$ *and*

$$P_n(x) = x^n + \cdots, \quad n = 1, 2, \ldots.$$

However, we will allow the first polynomial $P_0(x)$ to be any constant (not necessarily equal to 1).

A second solution of (17) is given by the so-called *associated orthogonal polynomials* $P_n^*(x)$ defined by [3, 4.3]

$$xP_n^* = P_{n+1}^* + \beta_n P_n^* + \gamma_n P_{n-1}^*, \quad P_0^* = 0, \quad P_1^* = 1. \tag{18}$$

Note that $\deg P_n^*(x) = n - 1$.

The converse of Theorem 9 is given by Favard's Theorem.

Theorem 11 *Let $\{P_n\}$ be a sequence of polynomials satisfying the recurrence relation*

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}, \quad n = 1, 2, \dots,$$

where $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C} \setminus \{0\}$. Then, there exists a unique linear functional $\mathcal{L}$ such that $\mathcal{L}[P_0] = 1$ and

$$\mathcal{L}[P_n P_m] = h_n \delta_{n,m},$$

with $h_0 = P_0$, $h_1 = \gamma_1$ and

$$h_n = \gamma_n h_{n-1}, \quad n = 2, 3, \dots. \tag{19}$$

Proof. We follow [3, Theorem 4.4] and define $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}[P_0] &= 1, \\ \mathcal{L}[P_n] &= 0, \quad n = 1, 2, \dots. \end{aligned}$$

Then,

$$\begin{aligned} h_0 &= \mathcal{L}\left[P_0^2\right] = P_0 \mathcal{L}[P_0] = P_0, \\ h_1 &= \mathcal{L}[xP_1] = \gamma_1, \end{aligned}$$

and

$$\mathcal{L}[xP_n] = 0, \quad n > 1.$$

Similarly, we find that for $n = 2, 3, \dots,$

$$\begin{aligned} \mathcal{L}\left[x^k P_n\right] &= 0, \quad 0 \le k < n \\ h_n = \mathcal{L}[x^n P_n] &= \mathcal{L}\left[x^{n-1} \gamma_n P_{n-1}\right] = \gamma_n h_{n-1}. \end{aligned}$$

■

Remark 12 *It follows from (4) and Favard's theorem that (at least for $k \neq 2$) $\{D_{n,k}\}$ is a sequence of monic orthogonal polynomials with respect to a moment functional L* [1] *satisfying*

$$L[D_{n,k} D_{m,k}] = h_n(k)\, \delta_{n,m}, \tag{20}$$

with

$$h_0(k) = 2 - k, \quad h_n(k) = a^n, \quad n = 1, 2, \dots. \tag{21}$$

In Section 4 we will find a representation for the moment functional L.

[1] In the remainder of the paper L will denote the moment functional associated with the polynomials $D_{n,k}(x; a)$.

Proposition 13 *Let $\mathcal{L}$ be a moment functional and $\{P_n\}$ be the sequence of monic orthogonal polynomials associated with it. Then, the following are equivalent:*

(a) All the moments of odd order are zero,

$$\mathcal{L}\left[x^{2n+1}\right] = 0, \quad n = 0, 1, \ldots.$$

(b)

$$P_n(-x) = (-1)^n P_n(x), \quad n = 0, 1, \ldots.$$

(c) All the coefficients β_n in the three-term recurrence relation (17) are zero

$$P_{n+1} = xP_n - \gamma_n P_{n-1}, \quad n = 1, 2, \ldots.$$

Proof. See [3, Theorem 4.3]. ■

Proposition 14 *Let $k \neq 2$ and $\mu_n(k)$ denote the moments of the linear functional defined by (20). Then, we have*

$$\mu_0(k) = \frac{1}{2-k}, \tag{22}$$

$$\mu_{2n+1}(k) = 0, \quad n = 0, 1, \ldots, \tag{23}$$

and

$$\mu_{2n} = -\sum_{l=0}^{n-1} \frac{(2-k)n + kl}{l+n} \binom{n+l}{2l} (-a)^{n-l} \mu_{2l}, \quad n = 1, 2, \ldots.$$

The first few nonzero moments are

$$\mu_2(k) = a, \quad \mu_4(k) = -a^2(k-3), \quad \mu_6(k) = a^3(k^2 - 6k + 10),$$
$$\mu_8(k) = -a^4(k^3 - 9k^2 + 29k - 35).$$

Proof. From (21), we see that

$$2 - k = h_0 = L\left[D_{0,k}^2\right] = D_{0,k}^2 L[1] = (2-k)^2 \mu_0,$$

from which (22) follows.

Using (1), it is clear that

$$D_{n,k}(-x; a) = (-1)^n D_{n,k}(x; a), \tag{24}$$

and Proposition 13 gives

$$\mu_{2n+1}(k) = 0, \quad n = 0, 1, \ldots.$$

From (6) we have

$$D_{2n,k}(x;a) = \sum_{l=0}^{n} \frac{(2-k)\,n + kl}{l+n} \binom{n+l}{2l} (-a)^{n-l} x^{2l},$$

and therefore

$$\begin{aligned} 0 = L\left[D_{2n,k}\right] &= \sum_{l=0}^{n} \frac{(2-k)\,n + kl}{l+n} \binom{n+l}{2l} (-a)^{n-l} \mu_{2l}(k) \\ &= \sum_{l=0}^{n-1} \frac{(2-k)\,n + kl}{l+n} \binom{n+l}{2l} (-a)^{n-l} \mu_{2l}(k) + \mu_{2n}(k). \end{aligned}$$

■

Remark 15 *In Section 4 we will find an explicit expression for* $\mu_{2n}(k)$.

The task of finding an explicit integral representation for the functional $\mathcal{L}$ is called a *moment problem* [1],[13],[17]. A moment functional $\mathcal{L}$ is called *determinate* if there exists a unique (up to an additive constant) distribution $\psi(t)$ such that

$$\mathcal{L}\left[x^n\right] = \int_{\Lambda} t^n d\psi. \tag{25}$$

Otherwise, $\mathcal{L}$ is called *indeterminate* [2], [18].

A criteria to decide if the moment functional $\mathcal{L}$ is determinate is due to Torsten Carleman [17, P 59]: If $\gamma_n > 0$ and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\gamma_n}} = \infty,$$

then $\mathcal{L}$ is determinate.

One method to find a distribution function satisfying (25) is given by Markov's theorem.

Theorem 16 *Let the moment functional $\mathcal{L}$ be determinate and $P_n(z)$ be the monic orthogonal polynomials with respect to $\mathcal{L}$ and $P_n^*(z)$ be the associated polynomials. Then, there exists a set $\Lambda \subset \mathbb{C}$ such that*

$$\lim_{n\to\infty} \mu_0 \frac{P_n^*(z)}{P_n(z)} = \int_{\Lambda} \frac{d\psi(t)}{z-t}, \quad z \notin \Lambda, \tag{26}$$

where μ_0 is the first moment of $\mathcal{L}$ and the convergence is uniform on compact subsets of $\mathbb{C} \setminus \Lambda$.

Proof. See [20]. ■

The function

$$\mathcal{S}(z) = \int_{\Lambda} \frac{d\psi(t)}{z-t}, \quad z \notin \Lambda, \tag{27}$$

is called the *Stieltjes transform* of $\psi(t)$ [19]. To recover the distribution function $\psi(t)$ from (27), we can use the *Stieltjes-Perron inversion formula [19, A.1.2]*

$$[\psi](s) - [\psi](t) = \frac{1}{\pi} \lim_{y\to 0^+} \int_s^t \operatorname{Im} \mathcal{S}(x + \mathrm{i}y)\, dx, \tag{28}$$

where $[\psi]$ denotes the jump operator

$$[\psi](s) = \lim_{\varepsilon\to 0^+} \frac{\psi(s+\varepsilon) + \psi(s-\varepsilon)}{2}.$$

The absolutely continuous part of ψ is given by

$$\psi'(t) = -\frac{1}{\pi} \lim_{y\to 0^+} \operatorname{Im} \mathcal{S}(t + \mathrm{i}y).$$

The function $\mathcal{S}(z)$ has the asymptotic behavior [10, Section 12.9]

$$\mathcal{S}(z) \sim \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots, \quad z \to \infty. \tag{29}$$

It is clear from the three-term recurrence relation (4) that the polynomials $D_{n,k}(x;a)$ are related to the Chebyshev polynomials. In the next section, we review their main properties.

3.1 Chebyshev polynomials

The Chebyshev polynomials of the first kind $T_n(x)$ are defined by [11, 9.8.35]

$$T_n(x) = {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} ; \frac{1-x}{2} \right),$$

and the Chebyshev polynomials of the second kind $U_n(x)$ are defined by [11, 9.8.36]

$$U_n(x) = (n+1)\ {}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix} ; \frac{1-x}{2} \right).$$

They are independent solutions of the recurrence relation

$$y_{n+1} - 2xy_n + y_{n-1} = 0, \tag{30}$$

with initial conditions [11, 9.8.39]

$$T_0(x) = 1, \quad T_1(x) = x, \tag{31}$$

and [11, 9.8.40]

$$U_0(x) = 1, \quad U_1(x) = 2x. \tag{32}$$

Using (32) in (30) we note that

$$U_{-1}(x) = 0. \tag{33}$$

Their ordinary generating functions are [11, 9.8.50]

$$\sum_{n=0}^{\infty} T_n(x) z^n = \frac{1-xz}{1-2xz+z^2}, \tag{34}$$

and [11, 9.8.56]

$$\sum_{n=0}^{\infty} U_n(x) z^n = \frac{1}{1-2xz+z^2}. \tag{35}$$

From (33) and

$$\begin{aligned} \sum_{n=0}^{\infty} \left[U_n(x) - xU_{n-1}(x)\right] z^n &= \sum_{n=0}^{\infty} U_n(x) z^n - x \sum_{n=-1}^{\infty} U_n(x) z^{n+1} \\ &= \frac{1-xz}{1-2xz+z^2}, \end{aligned}$$

we see that

$$T_n(x) = U_n(x) - xU_{n-1}(x), \quad n = 0, 1, \ldots. \tag{36}$$

The Chebyshev polynomials satisfy the orthogonality relations [11, 9.8.37]

$$\frac{1}{\pi}\int_{-1}^{1} T_n(t)\, T_m(t)\, \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}(1+\delta_{n,0})\,\delta_{n,m} \tag{37}$$

and [11, 9.8.38]

$$\frac{1}{\pi}\int_{-1}^{1} U_n(t)\, U_m(t)\, \sqrt{1-t^2}dt = \frac{1}{2}\delta_{n,m}. \tag{38}$$

The Stieltjes transforms of their distributions are given by [19]

$$\frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\frac{dt}{z-t} = \frac{1}{z\sqrt{1-z^{-2}}}, \quad z \in \mathbb{C}\setminus[-1,1], \tag{39}$$

and

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{z-t}dt = z\left(1-\sqrt{1-z^{-2}}\right), \quad z \in \mathbb{C}\setminus[-1,1], \tag{40}$$

where here and in the rest of the paper

$$\sqrt{} : \mathbb{C} \to \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2} \right\}$$

denotes the principal branch of the square root. Note that

$$\sqrt{1-z^{-2}} \sim 1, \quad z \to \infty.$$

From (32) and (33) we see that the associated polynomials $T_n^*(x), U_n^*(x)$ are

$$T_n^*(x) = U_n^*(x) = U_{n-1}(x).$$

Using Markov's theorem (26), we conclude that

$$\lim_{n\to\infty} \frac{U_{n-1}(z)}{T_n(z)} = \frac{1}{z\sqrt{1-z^{-2}}}, \quad z \in \mathbb{C}\setminus[-1,1], \tag{41}$$

and

$$\lim_{n\to\infty} \frac{U_{n-1}(z)}{U_n(z)} = z\left(1-\sqrt{1-z^{-2}}\right), \quad z \in \mathbb{C}\setminus[-1,1]. \tag{42}$$

Remark 17 *Note that Markov's theorem (26) refers to monic orthogonal polynomials and the first moment* μ_0 *appears in the limit formula. However, since the monic Chebyshev polynomials* $\widehat{T}_n(x)$, $\widehat{U}_n(x)$ *are given by* $\widehat{T}_0(x) = 1$ *and*

$$\widehat{T}_n(x) = 2^{1-n} T_n(x), \quad n = 1, 2, \ldots,$$
$$\widehat{U}_n(x) = 2^{-n} U_n(x), \quad n = 0, 1, \ldots,$$

the first moment $\mu_0 = \frac{1}{2}$ *is cancelled in (42).*

We can now relate the polynomials $D_{n,k}(x;a)$ and the Chebyshev polynomials.

Proposition 18 *The Dickson polynomials of the* $(k+1)$*-th kind can be written as*

$$D_{n,k}(x;a) = a^{\frac{n}{2}} \left[2(1-k) T_n\left(\frac{x}{2\sqrt{a}}\right) + kU_n\left(\frac{x}{2\sqrt{a}}\right) \right]. \tag{43}$$

Proof. Comparing the generating functions (34)-(35) with the generating function of $D_{n,k}(x;a)$ (16), we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a^{\frac{n}{2}} \left[2(1-k) T_n\left(\frac{x}{2\sqrt{a}}\right) + kU_n\left(\frac{x}{2\sqrt{a}}\right) \right] z^n \\
&= 2(1-k) \frac{1 - \frac{x}{2\sqrt{a}} z\sqrt{a}}{1 - 2\frac{x}{2\sqrt{a}} z\sqrt{a} + az^2} + k \frac{1}{1 - 2\frac{x}{2\sqrt{a}} z\sqrt{a} + az^2} \\
&= \frac{2 - k + (k-1)xz}{az^2 - xz + 1} = \sum_{n=0}^{\infty} D_{n,k}(x;a) z^n,
\end{aligned}$$

and the result follows. ■

Remark 19 *If we use the values of the Chebyshev polynomials at* $x = 0$ *[16, 18.6.1]*

$$T_n(0) = \cos\left(\frac{n\pi}{2}\right) = U_n(0),$$

and the representation (43), we get

$$D_{n,k}(0;a) = (2-k)\, a^{\frac{n}{2}} \cos\left(\frac{n\pi}{2}\right),$$

in agreement with (6)-(7).

If $k=0,1$, (43) gives

$$D_{n,0}(x;a) = 2a^{\frac{n}{2}} T_n\left(\frac{x}{2\sqrt{a}}\right),$$
$$D_{n,1}(x;a) = a^{\frac{n}{2}} U_n\left(\frac{x}{2\sqrt{a}}\right),$$

and in particular, for $a=1$, we have

$$D_n(2x) = D_{n,0}(2x;1) = 2T_n(x),$$
$$E_n(2x) = D_{n,1}(2x;1) = U_n(x),$$

as it was observed in [21]. For $k=2$, we obtain the following result.

Corollary 20 *We have*

$$D_{n,2}(x;a) = a^{\frac{1}{2}(n-1)} x U_{n-1}\left(\frac{x}{2\sqrt{a}}\right). \tag{44}$$

Proof. Setting $k=2$ in (43), we get

$$D_{n,2}(x;a) = 2a^{\frac{n}{2}}\left[U_n\left(\frac{x}{2\sqrt{a}}\right) - T_n\left(\frac{x}{2\sqrt{a}}\right)\right]$$

and using (36) we obtain

$$D_{n,2}(x;a) = 2a^{\frac{n}{2}}\frac{x}{2\sqrt{a}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right).$$

■

Remark 21 *A representation of $D_{n,2}(x;a)$ in terms of associated Legendre functions of the first and second kinds [16, 14.3] was given in "A representation of the Dickson polynomials of the third kind by Legendre functions" by Neranga Fernando and Solomon Manukure (arXiv:1604.04682).*

Lemma 22 *Let $D_n^*(x;a)$ denote the associated Dickson polynomials of the $(k+1)$-th kind satisfying the recurrence (4) with initial conditions*

$$D_0^*(x;a) = 0, \quad D_1^*(x;a) = 1.$$

Then,

$$D_n^*(x;a) = a^{\frac{n-1}{2}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right). \tag{45}$$

Proof. Let the polynomials $y_n(x)$ be defined by

$$y_n(x) = a^{\frac{n-1}{2}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right).$$

Then,

$$y_{n+1} - xy_n + ay_{n-1} = a^{\frac{n}{2}} U_n\left(\frac{x}{2\sqrt{a}}\right)$$
$$-xa^{\frac{n-1}{2}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right) + aa^{\frac{n-2}{2}} U_{n-2}\left(\frac{x}{2\sqrt{a}}\right).$$

Changing variables to $x = 2\sqrt{a}s$ and using (30) we get

$$y_{n+1} - xy_n + ay_{n-1} = a^{\frac{n}{2}} \left[U_n(s) - 2sU_{n-1}(s) + U_{n-2}(s)\right] = 0.$$

Finally, using (32) and (33) we have

$$y_0(x) = a^{-\frac{1}{2}} U_{-1}\left(\frac{x}{2\sqrt{a}}\right) = 0$$
$$y_1(x) = U_0\left(\frac{x}{2\sqrt{a}}\right) = 1.$$

Hence, $y_n(x) = D_n^*(x; a)$. ■

4 Main results

4.1 The scaled polynomials

Let's introduce the scaled polynomials $d_n(x;k)$ defined by

$$d_n(x;k) = a^{-\frac{n}{2}} D_{n,k}(2\sqrt{a}x; a). \tag{46}$$

The polynomials $d_n(x;k)$ are a solution of the same recurrence (30) satisfied by the Chebyshev polynomials, with initial conditions

$$d_0(x;k) = 2 - k, \quad d_1(x;k) = 2x.$$

From (43) it follows that

$$d_n(x;k) = 2(1-k) T_n(x) + kU_n(x), \tag{47}$$

and from (45) we have

$$d_n^*(x) = U_{n-1}\left(x\right) = a^{\frac{1-n}{2}} D_n^*(2\sqrt{a}x; a), \tag{48}$$

where $d_n^*(x)$ denote the associated polynomials satisfying (30) with initial conditions

$$d_0^*(x) = 0, \quad d_1^*(x) = 1.$$

Using (47) and (48) we can find the Stieltjes transform of the distribution associated with the polynomials $d_n(x; k)$.

Proposition 23 *Let $z \in \mathbb{C} \setminus [-1, 1]$ and*

$$\omega\left(k\right) = \frac{1}{2}\frac{k-2}{\sqrt{k-1}}\mathrm{i}. \tag{49}$$

Then,

$$\lim_{n\to\infty} \frac{d_n^*(z)}{d_n(z; k)} = s(z; k), \tag{50}$$

where

$$s(z; k) = z\frac{(k-2)\sqrt{1-z^{-2}}+k}{4\left(k-1\right)z^2+\left(k-2\right)^2} \tag{51}$$

and $z \neq \pm\omega$ if $k \neq 1$.

Proof. Let $z \in \mathbb{C} \setminus [-1, 1]$. From (41)-(42), we have

$$\lim_{n\to\infty} \frac{T_n\left(z\right)}{U_{n-1}(z)} = z\sqrt{1-z^{-2}},$$

and

$$\lim_{n\to\infty} \frac{U_n(z)}{U_{n-1}(z)} = \frac{1}{z\left(1-\sqrt{1-z^{-2}}\right)} = z\left(1+\sqrt{1-z^{-2}}\right). \tag{52}$$

Using (47) and (48), we get

$$\lim_{n\to\infty} \frac{d_n(z; k)}{d_n^*(z)} = 2\left(1-k\right)z\sqrt{1-z^{-2}} + kz\left(1+\sqrt{1-z^{-2}}\right).$$

If $k \neq 1$ and $z \neq \pm\omega$,we obtain

$$\lim_{n\to\infty} \frac{d_n^*(z)}{d_n(z; k)} = \frac{1}{2\left(1-k\right)z\sqrt{1-z^{-2}} + kz\left(1+\sqrt{1-z^{-2}}\right)}$$
$$= z\frac{(k-2)\sqrt{1-z^{-2}}+k}{4\left(k-1\right)z^2+\left(k-2\right)^2}.$$

If $k=1$, we see from (47) that

$$d_n(z;1) = U_n(z),$$

and using (52) we obtain

$$\lim_{n\to\infty} \frac{d_n^*(z)}{d_n(z;1)} = \frac{1}{z\left(1+\sqrt{1-z^{-2}}\right)}$$
$$= z\left(1-\sqrt{1-z^{-2}}\right) = s(z;1).$$

■

Our next objective is to represent the function $s(z;k)$ as the Stieltjes transform of a distribution. We begin with a couple of lemmas.

Lemma 24 *Let $z,b \in \mathbb{C}\setminus[-1,1]$. Then,*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-b^2}\frac{1}{z-t}dt = z\frac{\sqrt{1-b^{-2}}-\sqrt{1-z^{-2}}}{z^2-b^2}. \tag{53}$$

Proof. We can write

$$\frac{1}{(t^2-b^2)(z-t)} = \frac{1}{(z^2-b^2)(z-t)}$$
$$+\frac{1}{2b(b-z)(b-t)} + \frac{1}{2b(b+z)(-b-t)}.$$

Using (40), we obtain

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-b^2}\frac{1}{z-t}dt = \frac{z\left(1-\sqrt{1-z^{-2}}\right)}{z^2-b^2} + \frac{b\left(1-\sqrt{1-b^{-2}}\right)}{2b(b-z)}$$
$$-\frac{b\left(1-\sqrt{1-b^{-2}}\right)}{2b(b+z)}, \quad z,b\in\mathbb{C}\setminus[-1,1],$$

and the result follows. ■

Lemma 25 *Let $\omega(k)$ be defined by (49). Then,*

$$\omega(k)\in\mathbb{C}\setminus[-1,1], \quad k\in\mathbb{R}\setminus\{0,1,2\}.$$

Proof. The result follows immediately from the definition

$$\omega(k) = \frac{1}{2}\frac{k-2}{\sqrt{k-1}}\mathrm{i},$$

since

$$\begin{aligned}
\omega(k) &\in (-\infty, -1), \quad k \in (-\infty, 0), \\
\omega(0) &= -1, \\
\omega(k) &\in (-\infty, -1), \quad k \in (0, 1), \\
-\mathrm{i}\omega(k) &\in (-\infty, 0), \quad k \in (1, 2), \\
\omega(2) &= 0, \\
-\mathrm{i}\omega(k) &\in (0, \infty), \quad k \in (2, \infty).
\end{aligned}$$

■

The function $s(z;k)$ has a branch cut on the segment $[-1,1]$ and (perhaps removable) poles at $z = \pm\omega$ if $k \neq 1$. In the next proposition we split $s(z;k)$ in two parts, one analytic in $\mathbb{C} \setminus [-1,1]$ and the other analytic in $\mathbb{C} \setminus \{\pm\omega\}$.

Proposition 26 *Let $k \neq 2$ and $z \in \mathbb{C} \setminus [-1,1]$, with $z \neq \pm\omega$ if $k \neq 1$. Then, we have*

$$\frac{s(z;k)}{2-k} = s_c(z;k) + \chi(k)\, s_d(z;k), \tag{54}$$

where $\chi(k)$ is the characteristic function defined by

$$\chi(k) = \begin{cases} 0, & k \in [0,2] \\ 1, & k \in \mathbb{R} \setminus [0,2] \end{cases}, \tag{55}$$

$$s_c(z;k) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{4(k-1)t^2 + (k-2)^2}\frac{1}{z-t}dt, \tag{56}$$

$$s_d(z;k) = \frac{2k}{2-k}\frac{z}{4(k-1)z^2 + (k-2)^2}, \tag{57}$$

and $s(z;k)$ was defined in (51).

Proof. Let $k \in \mathbb{R} \setminus \{0,1,2\}$. From (51) we have

$$\frac{s(z;k)}{2-k} = \frac{z}{4(k-1)}\frac{-\sqrt{1-z^{-2}} + \frac{k}{2-k}}{z^2 + \frac{(k-2)^2}{4(k-1)}},$$

or

$$\frac{s(z;k)}{2-k} = -\frac{z}{4\left(k-1\right)} \frac{\sqrt{1-z^{-2}}+\frac{k}{k-2}}{z^2-\omega^2}.$$

Since we know that $\omega\left(k\right) \in \mathbb{C}\setminus[-1,1]$ from Lemma 25, we can use (53) with $b=\omega$ and obtain

$$\begin{aligned} \frac{s(z;k)}{2-k} &= \frac{1}{4\left(k-1\right)} \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-\omega^2} \frac{1}{z-t} dt \\ &\quad - \frac{1}{4\left(k-1\right)} \left(\sqrt{1-\omega^{-2}} + \frac{k}{k-2} \right) \frac{z}{z^2-\omega^2}. \end{aligned}$$

But we have

$$1-\omega^{-2} = \frac{k^2}{\left(k-2\right)^2} \tag{58}$$

and therefore

$$\begin{aligned} \sqrt{1-\omega^{-2}} + \frac{k}{k-2} &= \left| \frac{k}{k-2} \right| + \frac{k}{k-2} \\ &= \begin{cases} 0, & k \in [0,2) \\ \frac{2k}{k-2}, & k \in \mathbb{R}\setminus[0,2] \end{cases}. \end{aligned}$$

Thus, for $k \in (0,2)\setminus\{1\}$ we get

$$\frac{s(z;k)}{2-k} = \frac{1}{4\left(k-1\right)} \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-\omega^2} \frac{1}{z-t} dt,$$

and for $k \in \mathbb{R}\setminus[0,2]$

$$\begin{aligned} \frac{s(z;k)}{2-k} &= \frac{1}{4\left(k-1\right)} \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-\omega^2} \frac{1}{z-t} dt \\ &\quad + \frac{2k}{2-k} \frac{1}{4\left(k-1\right)} \frac{z}{z^2-\omega^2}. \end{aligned}$$

If $k=0$, then we have from (51)

$$s(z;0) = \frac{z}{2} \frac{\sqrt{1-z^{-2}}}{z^2-1} = \frac{1}{2z\sqrt{1-z^{-2}}}$$

and using (39) we get

$$\frac{s(z;0)}{2} = \frac{1}{4\pi}\int\limits_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \frac{dt}{z-t}$$

$$= \frac{1}{\pi}\int\limits_{-1}^{1} \frac{\sqrt{1-t^2}}{4\,(1-t^2)} \frac{1}{z-t} dt = s_c(z;0).$$

If $k=1$, then we have from (51)

$$s(z;1) = z\left(1-\sqrt{1-z^{-2}}\right)$$

and using (40) we get

$$\frac{s(z;1)}{1} = \frac{1}{\pi}\int\limits_{-1}^{1} \frac{\sqrt{1-t^2}}{z-t} dt = s_c(z;1).$$

■

Remark 27 *Note that the function $s_c(z;k)$ is analytic for $z \in \mathbb{C} \setminus [-1,1]$, since it is the Stieltjes transform of a distribution [19, A.1]*

$$s_c(z;k) = \frac{1}{4\,(k-1)} \frac{1}{\pi}\int\limits_{-1}^{1} \frac{\sqrt{1-t^2}}{t^2-\omega^2} \frac{1}{z-t} dt.$$

In fact, (53) gives

$$s_c(z;k) = \frac{z}{4\,(k-1)} \frac{\sqrt{1-\omega^{-2}}-\sqrt{1-z^{-2}}}{z^2-\omega^2}, \quad k \neq 1 \tag{59}$$

which is analytic at $z = \pm\omega$ with

$$\lim_{z\to\pm\omega} s_c(z;k) = \pm\frac{1}{4\,(k-1)} \frac{\sqrt{1-\omega^{-2}}}{2\omega\,(1-\omega^2)}, \quad k \neq 0,1.$$

For $k=0$, we have

$$s_c(z;0) = \frac{z}{4} \frac{\sqrt{1-z^{-2}}}{z^2-1} = \frac{1}{4} \frac{1}{\sqrt{1-z^{-2}}},$$

since $\omega(0) = -1$.

Finally, rewriting (59) as

$$s_c(z;k) = z\frac{\sqrt{1-\omega^{-2}} - \sqrt{1-z^{-2}}}{4(k-1)z^2 + (k-2)^2},$$

we see that

$$s_c(z;1) = z\left(\sqrt{1-\omega^{-2}} - \sqrt{1-z^{-2}}\right) = z\left(1 - \sqrt{1-z^{-2}}\right),$$

since $\omega^{-2} \to 0$ *as* $k \to 1$.

4.2 The Dickson polynomials

We can now apply the previous results to the Dickson polynomials of the $(k+1)$-th kind $D_{n,k}(x;a)$, related to the scaled polynomials $d_n(x;k)$ by (46).

Proposition 28 *Let* $z \in \mathbb{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$ *and*

$$\Omega(k,a) = (k-2)\sqrt{\frac{a}{k-1}}\mathrm{i}. \tag{60}$$

Then,

$$\lim_{n\to\infty} \frac{D_n^*(z;a)}{D_{n,k}(z;a)} = S(z;k,a),$$

where

$$S(z;k,a) = \frac{z}{2}\frac{(k-2)\sqrt{1-4az^{-2}} + k}{(k-1)z^2 + a(k-2)^2} \tag{61}$$

and $z \neq \pm\Omega$ *if* $k \neq 1$.

Proof. Using (46), (48) and (50), we have

$$\lim_{n\to\infty} \frac{D_n^*(z;a)}{D_{n,k}(z;a)} = \lim_{n\to\infty} \frac{a^{\frac{n-1}{2}} d_n^*\left(\frac{z}{2\sqrt{a}}\right)}{a^{\frac{n}{2}} d_n\left(\frac{z}{2\sqrt{a}};k\right)} = \frac{1}{\sqrt{a}} s\left(\frac{z}{2\sqrt{a}};k\right), \tag{62}$$

as long as

$$\frac{z}{2\sqrt{a}} \notin [-1,1],$$
$$\frac{z}{2\sqrt{a}} \neq \pm\omega \quad \text{if} \quad k \neq 1.$$

Also,

$$\frac{1}{\sqrt{a}} s\left(\frac{z}{2\sqrt{a}}; k\right) = \frac{z}{2a} \frac{(k-2)\sqrt{1-\left(\frac{z}{2\sqrt{a}}\right)^{-2}}+k}{4(k-1)\left(\frac{z}{2\sqrt{a}}\right)^{2}+(k-2)^{2}}$$
$$= \frac{z}{2} \frac{(k-2)\sqrt{1-4az^{-2}}+k}{(k-1)z^{2}+a(k-2)^{2}}.$$

■

We finally have all the necessary elements to find an explicit representation for the moment functional L associated with the polynomials $D_{n,k}(x;a)$.

Theorem 29 *Let $k \neq 2$ and $L : \mathbb{C}[x] \to \mathbb{C}$ be the linear functional defined by (20). Then, L admits the representation*

$$L[q] = L_c[q] + \chi(k) L_d[q],$$

where $\chi(k)$ was defined in (55),

$$L_c[q] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} q(t) \frac{\sqrt{4a-t^2}}{(k-1)t^2+a(k-2)^2} dt \tag{63}$$

and

$$L_d[q] = \frac{k}{2(k-1)(2-k)} [q(\Omega)+q(-\Omega)], \tag{64}$$

with Ω defined by (60).

Proof. From (22) we know that the first moment of L is given by

$$\mu_0 = \frac{1}{2-k}.$$

Thus, from Markov's theorem (26) we have

$$\lim_{n\to\infty} \frac{1}{2-k} \frac{D_n^*(z;a)}{D_{n,k}(z;a)} = \int_{\Lambda_k} \frac{d\psi(t;k)}{z-t}, \quad z \in \mathbb{C} \backslash \Lambda_k.$$

From (62) we get

$$\lim_{n\to\infty}\frac{1}{2-k}\frac{D_n^*(z;a)}{D_{n,k}(z;a)}=\frac{1}{\sqrt{a}}\frac{1}{2-k}s\left(\frac{z}{2\sqrt{a}};k\right)$$

and from (54) we obtain

$$\lim_{n\to\infty}\frac{1}{2-k}\frac{D_n^*(z;a)}{D_{n,k}(z;a)}=\frac{1}{\sqrt{a}}\left[s_c\left(\frac{z}{2\sqrt{a}};k\right)+\chi\left(k\right)s_d\left(\frac{z}{2\sqrt{a}};k\right)\right].$$

Using (56) we have

$$\frac{1}{\sqrt{a}}s_c\left(\frac{z}{2\sqrt{a}};k\right)=\frac{1}{\pi\sqrt{a}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{4\left(k-1\right)t^2+\left(k-2\right)^2}\frac{dt}{\frac{z}{2\sqrt{a}}-t},$$

as long as $\frac{z}{2\sqrt{a}}\notin[-1,1]$ (see Remark 27). Changing variables to $t=\frac{\tau}{2\sqrt{a}}$, we get

$$\begin{aligned}\frac{1}{\sqrt{a}}s_c\left(\frac{z}{2\sqrt{a}};k\right)&=\frac{1}{2\pi a}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{\sqrt{1-\frac{\tau^2}{4a}}}{4\left(k-1\right)\frac{\tau^2}{4a}+\left(k-2\right)^2}\frac{d\tau}{\frac{z}{2\sqrt{a}}-\frac{\tau}{2\sqrt{a}}}\\&=\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{\sqrt{4a-\tau^2}}{\left(k-1\right)\tau^2+a\left(k-2\right)^2}\frac{d\tau}{z-\tau}.\end{aligned}$$

Using (56) we have

$$\begin{aligned}\frac{1}{\sqrt{a}}s_d\left(\frac{z}{2\sqrt{a}};k\right)&=\frac{1}{\sqrt{a}}\frac{2k}{2-k}\frac{z}{2\sqrt{a}}\frac{1}{4\left(k-1\right)\frac{z^2}{4a}+\left(k-2\right)^2}\\&=\frac{k}{2-k}\frac{z}{\left(k-1\right)z^2+a\left(k-2\right)^2},\end{aligned}$$

as long as

$$\begin{aligned}&\frac{z}{2\sqrt{a}}\notin\left[-1,1\right],\\&\frac{z}{2\sqrt{a}}\neq\pm\omega\quad\text{if}\quad k\neq1.\end{aligned}$$

If $k \neq 1$, we can write

$$\frac{1}{\sqrt{a}} s_d \left(\frac{z}{2\sqrt{a}}; k \right) = \frac{k}{(2-k)(k-1)} \frac{z}{z^2 + a \frac{(k-2)^2}{(k-1)}}$$
$$= \frac{k}{(2-k)(k-1)} \frac{z}{z^2 - \Omega^2}$$

and using partial fractions,

$$\frac{1}{\sqrt{a}} s_d \left(\frac{z}{2\sqrt{a}}; k \right) = \frac{k}{2(2-k)(k-1)} \left(\frac{1}{z-\Omega} + \frac{1}{z+\Omega} \right),$$

with $z \neq \pm\Omega$.

Therefore, we conclude that

$$d\psi(t;k) = \frac{1}{2\pi} \frac{\sqrt{4a - \tau^2}}{(k-1)\tau^2 + a(k-2)^2} dt = d\psi_c(t;k), \quad k \in [0,2),$$
$$d\psi(t;k) = d\psi_c(t;k) + k \frac{\delta(t-\Omega) + \delta(t+\Omega)}{2(2-k)(k-1)} dt, \quad k \in \mathbb{R} \setminus [0,2],$$

where $\delta(t - t_0)$ denotes the Dirac delta function and

$$\Lambda_k = \begin{cases} [-2\sqrt{a}, 2\sqrt{a}], & k \in [0,2) \\ [-2\sqrt{a}, 2\sqrt{a}] \cup \{-\Omega, \Omega\}, & k \in \mathbb{R} \setminus [0,2] \end{cases}.$$

■

Remark 30 *We get the same result if we apply the Stieltjes-Perron inversion formula (28) to the function $S(z;k,a)$.*

Although Theorem 29 seems to be valid only when $k \neq 2$, we can find a valid interpretation even in this case.

Lemma 31 *Let $k \neq 1$ and Ω be defined by (60). Then,*

$$D_{n,k}(\Omega; a) = (2-k) \left(-\mathrm{i} \sqrt{\frac{a}{k-1}} \right)^n \tag{65}$$

Proof. Lets assume that

$$D_{n,k}\left(\Omega;a\right)=b_0B^n,$$

for some functions $b_0\left(k,a\right)$ and $B\left(k,a\right)$. Using (4), we have

$$0=b_0B^{n+2}-\Omega b_0B^{n+1}+ab_0B^n=b_0B^n\left(B^2-\Omega B+a\right).$$

Using (2), we get

$$b_0=D_{0,k}(\omega;a)=2-k$$

and

$$\Omega=D_{1,k}\left(\Omega;a\right)=\left(2-k\right)B\left(k,a\right).$$

Thus,

$$B\left(k,a\right)=\frac{\Omega}{2-k}=-\sqrt{\frac{a}{k-1}}\mathrm{i},$$

and clearly

$$B^2-\Omega B+a=0.$$

■

It follows from the previous Lemma that the discrete part of L is well defined when $k=2$.

Proposition 32 *Let Ω be defined by (60) and L_d be defined by (64). Then, for $k\neq 1$*

$$L_d\left[D_{n,k}D_{m,k}\right]=\left[\frac{1+\left(-1\right)^{n+m}}{2}\right]\frac{\left(2-k\right)k}{k-1}\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}.$$

Proof. From (65) we have

$$D_{n,k}\left(\Omega;a\right)D_{m,k}\left(\Omega;a\right)=\left(2-k\right)^2\left(-\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}.$$

Using (24), we get

$$\begin{aligned}&D_{n,k}\left(\Omega;a\right)D_{m,k}\left(\Omega;a\right)+D_{n,k}\left(-\Omega;a\right)D_{m,k}\left(-\Omega;a\right)\\&=\left(2-k\right)^2\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}\left[1+\left(-1\right)^{n+m}\right].\end{aligned}$$

Thus,

$$L_d\left[D_{n,k}D_{m,k}\right]=\left[\frac{1+(-1)^{n+m}}{2}\right]\frac{(2-k)\,k}{k-1}\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}.$$

■

We can now extend Theorem 29 to all values of k.

Corollary 33 *Let $h_n(k)$ be defined by (21) and $\chi(k)$ be defined by (55). Then,*

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{\sqrt{4a-t^2}D_{n,k}(t)\,D_{m,k}(t)}{(k-1)\,t^2+a\,(k-2)^2}dt \tag{66}$$

$$+\chi(k)\left[\frac{1+(-1)^{n+m}}{2}\right]\frac{(2-k)\,k}{k-1}\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}=h_n(k)\,\delta_{n,m}.$$

Remark 34 *If we set $k=2$ in (66), we obtain*

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{\sqrt{4a-t^2}}{t^2}D_{n,2}(t)\,D_{m,2}(t)\,dt=h_n(k)\,\delta_{n,m}, \tag{67}$$

which seems to make no sense, since the integrand is singular at $t=0$. However, if we use (44) we have

$$D_{n,2}(t;a)=a^{\frac{1}{2}(n-1)}tU_{n-1}\left(\frac{t}{2\sqrt{a}}\right),$$

and we can write (67) as

$$a^{\frac{1}{2}(n+m)-1}\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\sqrt{4a-t^2}U_{n-1}\left(\frac{t}{2\sqrt{a}}\right)U_{m-1}\left(\frac{t}{2\sqrt{a}}\right)dt=h_n(k)\,\delta_{n,m},$$

or changing variables to $\tau=\frac{t}{2\sqrt{a}}$

$$a^{\frac{n+m}{2}}\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}U_{n-1}(\tau)\,U_{m-1}(\tau)\,d\tau=h_n(k)\,\delta_{n,m}.$$

This agrees with (38), since we have $U_{-1} = 0$ from (33) and $h_0(2) = 0$ from (21), while for $n, m \geq 1$ we know from (21) that

$$a^{-\frac{n+m}{2}} h_n(k)\, \delta_{n,m} = \delta_{n,m}.$$

Finally, we will use the function $S(z; k, a)$ to find explicit expressions for the moments of L.

Proposition 35 *Let L be the linear functional defined by (20). Then, the moments of L of even order*

$$\mu_{2n}(k) = L\left[x^{2n}\right],$$

are given by

$$\mu_{2n}(1) = 2^{2n+1} \binom{\frac{1}{2}}{n+1} (-a)^n, \quad n = 0, 1, \ldots, \tag{68}$$

$$\mu_{2n}(2) = -2^{2n-1} \binom{\frac{1}{2}}{n} (-a)^n, \quad n = 1, 2, \ldots, \tag{69}$$

and if $k \neq 1, 2$,

$$\mu_{2n}(k) = -\frac{1}{2} \frac{(k-2)^{2n}}{(k-1)^{n+1}} (-a)^n \left(\frac{k}{k-2} + \sum_{l=0}^{n} \binom{\frac{1}{2}}{l} \left[\frac{4(k-1)}{(k-2)^2} \right]^l \right). \tag{70}$$

Proof. From (29) and (61), we have

$$\sum_{l=0}^{\infty} \frac{\mu_l(k)}{z^{l+1}} = \frac{S(z; k, a)}{2-k} = -\frac{z}{2} \frac{\sqrt{1 - 4az^{-2}} + \frac{k}{k-2}}{(k-1) z^2 + a(k-2)^2}.$$

Using (23), we get

$$\sum_{l=0}^{\infty} \frac{\mu_{2l}(k)}{z^{2l}} = -\frac{1}{2} \frac{\sqrt{1 - 4az^{-2}} + \frac{k}{k-2}}{k - 1 + a(k-2)^2 z^{-2}}.$$

Letting $u = z^{-2}$, we see that

$$\sum_{l=0}^{\infty} \mu_{2l}(k)\, u^l = -\frac{1}{2} \frac{\sqrt{1 - 4au} + \frac{k}{k-2}}{k - 1 + a(k-2)^2 u},$$

and therefore

$$\sqrt{1-4au}+\frac{k}{k-2}=-2\left[k-1+a\left(k-2\right)^{2}u\right]\sum_{l=0}^{\infty}\mu_{2l}u^{l}$$
$$=-\sum_{l=0}^{\infty}2\left(k-1\right)\mu_{2l}u^{l}-\sum_{l=1}^{\infty}2a\left(k-2\right)^{2}\mu_{2(l-1)}u^{l}.$$

Since

$$\sqrt{1-4au}=\sum_{l=0}^{\infty}\binom{\frac{1}{2}}{l}\left(-4au\right)^{l},$$

we obtain

$$1+\frac{k}{k-2}=-2\left(k-1\right)\mu_{0},$$

and

$$\binom{\frac{1}{2}}{l}\left(-4a\right)^{l}=-2\left(k-1\right)\mu_{2l}-2a\left(k-2\right)^{2}\mu_{2(l-1)},\quad l=1,2,\ldots.$$

If $k=1$, we get

$$\binom{\frac{1}{2}}{l}\left(-4a\right)^{l}=-2a\mu_{2(l-1)},\quad l=1,2,\ldots,$$

or

$$\mu_{2n}\left(1\right)=2^{2n+1}\binom{\frac{1}{2}}{n+1}\left(-a\right)^{n},\quad n=0,1,\ldots.$$

If $k=2$, we have

$$\mu_{2n}\left(2\right)=-2^{2n-1}\binom{\frac{1}{2}}{n}\left(-a\right)^{n},\quad n=1,2,\ldots.$$

If $k\neq 1,2$, we set $y_{l}=\mu_{2l}$, and obtain the recurrence

$$y_{l+1}=-\frac{a\left(k-2\right)^{2}}{k-1}y_{l}-\frac{\left(-4a\right)^{l+1}}{2\left(k-1\right)}\binom{\frac{1}{2}}{l+1},$$

with

$$y_{0}=\frac{1}{2-k}.$$

As it is well known, the general solution of the initial value problem

$$y_{n+1} = c_n y_n + g_n, \quad y_{n_0} = y_0,$$

is [8, 1.2.4]

$$y_n = y_0 \prod_{j=n_0}^{n-1} c_j + \sum_{k=n_0}^{n-1} \left(g_k \prod_{j=k+1}^{n-1} c_j \right).$$

Thus,

$$y_n = \frac{1}{2-k} \left[-\frac{a\,(k-2)^2}{k-1} \right]^n - \sum_{l=0}^{n-1} \frac{(-4a)^{l+1}}{2\,(k-1)} \binom{\frac{1}{2}}{l+1} \left[-\frac{a\,(k-2)^2}{k-1} \right]^{n-l-1},$$

or

$$y_n = -\frac{1}{2\,(k-1)} \left[-\frac{a\,(k-2)^2}{k-1} \right]^n \left(\frac{k}{k-2} + \sum_{l=0}^{n} \binom{\frac{1}{2}}{l} \left[\frac{4\,(k-1)}{(k-2)^2} \right]^l \right)$$

and the result follows. ■

Remark 36 *If $k = 0$, we get from (70)*

$$\mu_{2n}(0) = \frac{(4a)^n}{2} \sum_{l=0}^{n} (-1)^l \binom{\frac{1}{2}}{l},$$

and using the identity [16, 26.3.10]

$$\sum_{l=0}^{n} (-1)^l \binom{\alpha}{l} = (-1)^n \binom{\alpha - 1}{n},$$

we obtain

$$\mu_{2n}(0) = 2^{2n-1} \binom{-\frac{1}{2}}{n} (-a)^n.$$

This agrees with (63), since

$$\mu_{2n}(0) = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{t^{2n}}{\sqrt{4a - t^2}} dt.$$

When $k=1$, we have from (63)

$$\mu_{2n}(1) = \frac{1}{2\pi a}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\sqrt{4a-t^2}dt,$$

and therefore (68) gives

$$\frac{1}{2\pi a}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\sqrt{4a-t^2}dt = 2^{2n+1}\binom{\frac{1}{2}}{n+1}(-a)^n,$$

which can be verified directly.

When $k=2$, we can write (see Remark 34)

$$\mu_{2n}(2) = \frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\frac{\sqrt{4a-t^2}}{t^2}dt = \frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2(n-1)}\sqrt{4a-t^2}dt,$$

where $n=1,2,\ldots$. Hence,

$$\mu_{2n}(2) = a\mu_{2(n-1)}(1) = a2^{2(n-1)+1}\binom{\frac{1}{2}}{n}(-a)^{n-1}, \quad n=1,2,\ldots,$$

in agreement with (69).

5 Conclusions

We have shown that the Dickson polynomials of the $(k+1)$-th kind defined by

$$D_{n,k}(x;a) = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{n-kj}{n-j}\binom{n-j}{j}(-a)^j x^{n-2j}$$

satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{\sqrt{4a-t^2}D_{n,k}(t)\,D_{m,k}(t)}{(k-1)\,t^2+a\,(k-2)^2}dt$$
$$+\chi(k)\left[\frac{1+(-1)^{n+m}}{2}\right]\frac{(2-k)\,k}{k-1}\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m} = h_n(k)\,\delta_{n,m},$$

where $a > 0$, $k \in \mathbb{R}$,

$$\chi(k) = \begin{cases} 0, & k \in [0,2] \\ 1, & k \in \mathbb{R} \setminus [0,2] \end{cases},$$

and

$$h_0(k) = 2 - k, \quad h_n(k) = a^n, \quad n = 1, 2, \ldots.$$

We hope that this work will outline some connections between finite fields and orthogonal polynomials, and that it would be of interest to researchers in both areas.

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