Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

How to avoid collision of 3D-realization for moving graphs

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DK-Report No. 2019-12 12 2019

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How to avoid collision of 3D-realization for moving graphs

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Abstract. We define a specific type of 3D-model for moving graphs which we call L-model, the collision graph and induced collision graph of a given moving graph. We prove that if the edge set of a moving graph can be partitioned into two parts, such that the induced collision graph of both parts have no cycle, then there exists a collision-free L-model of the given moving graph. We also give algorithms showing how to get this height arrangement. In particular, we prove that any Dixon-1 moving graph has a collision-free L-model. Also, we prove that Dixon-2 moving graph has no collision-free L-model.

1 Introduction

Given a graph $G = (V, E)$, if we parametrize the position of each vertex v in the plane as $v(t) = (x_v(t), y_v(t))$ such that the distance between $v_i(t)$ and $v_j(t)$ is a constant if $v_iv_j \in E$, then this graph is called a moving graph. Examples of moving graphs can be found in [4], [2].

For a connected moving graph M, a realization of an L-model of M consists of vertical sticks and horizontal sticks. There is a 1 to 1 correspondence between vertical sticks and the vertex set of M; similarly for horizontal sticks and the edge set of M. There are joint structures at both ends of horizontal sticks so that they can be connected with vertical sticks. Figure 1 shows a realization of an L-model.

Fig. 1. Here is an L-model of a moving graph $K_{3,3}$. The edges correspond to sticks with holes, and the vertices correspond to gray columns.

* The research was funded by the Austrian Science Fund (FWF): Grant no.W1214-N15, project DK9.

One observes collision at the time when a vertex hits an edge. The collisions depend on the choice of heights of the horizontal sticks. Note that edge intersection is possible in some moving graphs. We can avoid this situation simply by setting all edges (horizontal sticks) to different heights. In this paper, we give a criterion which, when fulfilled, allows to avoid collisions by choosing the heights of edges (horizontal sticks) properly in an L-model of a moving graph. With this criterion we prove there exists a collision-free L-model for every Dixon-1 moving graph. In addition, we prove that Dixon-2 moving graphs cannot be realized in any L-model corresponding to it.

A result [1] by Abel et al. shows that any polynomial curve can be traced by a non-crossing linkage. Our result differs from them in the sense that we generate a motion (Dixon1), not just trace a curve. Another relevant result [3] is by Gallet et al. They provide an algorithm which produces linkages realizing a planar motion without collision. Our work differs from them in the sense that we consider arbitrary moving graphs, not just those arise from their version of Kempe.

2 Problem statement

For a clear problem statement we need to introduce some definitions first.

Definition 1 (moving graph). *A moving graph is a pair $M = (G, F)$, where $G = (V, E)$ is a graph and $F = \{f_v : \mathbb{R} \to \mathbb{R}^2, t \mapsto (x_v(t), y_v(t)) | v \in V, \|f_{v_i}(t) - f_{v_j}(t)\|$ is a constant if $v_i v_j \in E\}$.*

Definition 2 (collision in a moving graph). *Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. Vertex $v_k \in V$ collides with edge $v_i v_j \in E$ if and only if the equation $\|f_{v_k}(t) - f_{v_i}(t)\| + \|f_{v_k}(t) - f_{v_j}(t)\| = \|f_{v_i}(t) - f_{v_j}(t)\|$ has solution(s) in $\mathbb{R}$, where $f_{v_k}, f_{v_i}, f_{v_j} \in F$ and then $(v_k, v_i v_j)$ is called a* ***collision pair*** *in M.*

Definition 3 (L-model). *Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. An L-model of M is a pair $L = (M, h)$, where $h : E \to \mathbb{Z}$ is a function assigning to each edge of G an integer height value.*

Definition 4 (collision-free L-model). *Let $L = (M, h)$ be an L-model, where $M = (G, F)$ and $G = (V, E)$. L is collision-free if and only if*

$$h(v_i v_j) \notin [\min_{v v_k \in E} h(v v_k), \max_{v v_k \in E} h(v v_k)]$$

for any collision pair $(v_k, v_i v_j)$ in M.

The problem we focus on in this paper is: **Given a moving graph M, how to find a collision-free L-model for it.**

3 Collision detection

We have a collision-detection program, the algorithm of which is described in Algorithm1.

For a moving graph $M = (G, F)$ where $G = (V, E)$, apply Algorithm 1 and then we get the collision information of M. We implement our algorithm in Mathematica [5].

Algorithm 1: collision-detection program

```
Input   : function set F of a moving graph M = (G, F), where G = (V, E)
Output  : Collision pairs in M
for i = 0 to |V|
    for j = 0 to |E|
        if is a collision pair(solve equation in Definition 2) in M
            print (v_i, e_j)
```

Example 1. Consider one example from the class of Dixon-1 moving graphs [7], denote it as M, $M = (G, F)$, $G = (V, E)$. It is a complete bipartite graph with 7 vertices. Two independent sets of V are $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$.

Functions in F are:

$f_1(t) = (\sin t, 0)$,
$f_2(t) = (\sqrt{1 + \sin^2 t}, 0)$,
$f_3(t) = (-\sqrt{2 + \sin^2 t}, 0)$,
$f_4(t) = (\sqrt{3 + \sin^2 t}, 0)$,
$f_5(t) = (0, \cos t)$,
$f_6(t) = (0, \sqrt{1 + \cos^2 t})$,
$f_7(t) = (0, -\sqrt{2 + \cos^2 t})$.

Apply the collision-detection program, we get the collision pairs in M:
$(1, 52), (1, 53), (1, 54), (2, 54), (5, 61), (5, 71)$.

4 Sufficient condition for the existence of a collision-free L-model

After collecting the collision information for the given moving graph, we want to describe these information in a nicer way.

Definition 5 (collision graph). *Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. The collision graph of M, denoted as graph C, is $C = (V_C, E_C)$, where $V_C = E$ and $\overrightarrow{e_i e_j} \in E_C$ if and only if at least one of the vertices of e_i collide(s) with edge e_j in M.*

Remark 1. Note that the collision graph is a directed graph.

Definition 6 (induced collision graph). *For a moving graph $M = (G, F)$, where $G = (V, E)$. $S \subset E$, the collision graph of M induced by S is $C[S]$, the subgraph of C induced by S, where C is the collision graph of M.*

Example 2. We continue with Example 1. From collision information collected in Example 1, we construct the collision graph C as is shown in Figure 2.

Definition 7 (partition condition). *Let $C = (V_C, E_C)$ be a directed graph. If there exists a bi-partition of V_C into V_1 and V_2 such that $C[V_1]$ and $C[V_2]$ both are acyclic, then we say C fulfills **partition condition**.*

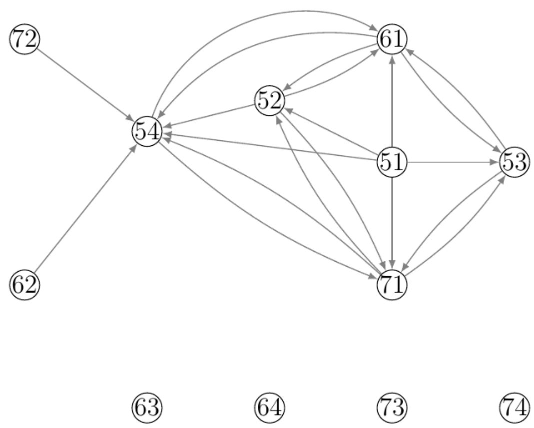

Fig. 2. Here is the collision graph of M - a Dixon-1 moving graph of $K_{3,4}$. A directed edge $(ij) \to (kl)$ means either (i, kl) or (j, kl) is a collision pair in M.

Theorem 1. *Let $M = (G, F)$ be a finite moving graph, where $G = (V, E)$. $C = (V_C, E_C)$ is the collision graph of M. If C fulfills partition condition then there exists a collision-free L-model of M.*

First we introduce two algorithms contructing the height function for the vertex set, given an acyclic directed graph.

Definition 8 (height order). *Given an acyclic directed graph U, the height order $<$ on the vertex set of U is defined as: $i < j$ if and only if there is a (directed) path from i to j.*

Remark 2. It is a strict weak order.

Proposition 1. *For a finite acyclic directed graph $U = (V, E)$, there exist at least one minimal element in V with respect to height order.*

Proof. If there is no minimal element in V under height order, then there is a infinite chain $v_1 > v_2 > ...$ in V. Since there is no cycle in U, so all the elements in this chain are pairwise distinct. This contradicts to the finiteness of $|V|$. □

Proof (termination of Algorithm 2 and Algorithm 3). If graph C is not empty, by Proposition 1 there must exist at least one minimal vertex under height order. So S is not be empty at the beginning. By "Step 2", the graph strictly reduces. Since the given graph is finite, both algorithms terminate. □

Proposition 2. *Let $M = (G, F)$ be a moving graph that fulfills the condition in Theorem 1 and two induced collision graphs are C_L and C_U. After applying Algorithm 2 to C_U, Algorithm 3 to C_L, we get a collision-free height function for an L-model of M.*

Algorithm 2: algorithm for constructing the height function for elements in V_C

Input : a finite acyclic directed graph $C = (V_C, E_C)$
Output : height values for V_C, here we denote height function as h

- Initialization: $k := 1$.
- Step 1: S:= collection of all minimal vertices in the vertex set of graph C under height order.
- Step 2: Replace C by $C[V_C - S]$, where $C[V_C - S]$ is the subgraph of C induced by $V_C - S$.
- Step 3: If $S = \emptyset$, go to Step 4; Otherwise, pick one element r in S, set $h(r) := k$. Replace S by $S - \{r\}$. $k := k + 1$, go to Step 3.
- Step 4: If $C = \emptyset$, return h; Otherwise, go to Step 1.

Algorithm 3: algorithm for constructing the height function for elements in V_C

Input : a finite acyclic directed graph $C = (V_C, E_C)$
Output : height values for V_C, here we denote height function as h

- Initialization: $k := 0$.
- Step 1: S:= collection of all minimal vertices in the vertex set of graph C under height order.
- Step 2: Replace C by $C[V_C - S]$, where $C[V_C - S]$ is the subgraph of C induced by $V_C - S$.
- Step 3: If $S = \emptyset$, go to Step 4; Otherwise, pick one element r in S, set $h(r) := k$. Replace S by $S - \{r\}$. $k := k - 1$, go to Step 3.
- Step 4: If $C = \emptyset$, return h; Otherwise, go to Step 1.

Proof. We start our consideration from graph $M = (G, F)$ where $G = (V, E)$. First divide its edges into two parts, E_U and E_L. Now we consider part E_U (the upper part, also the vertices of C_U): For any edge i (of graph G) in E_U, for all the vertices that collide with i, there is a direct edge from those edges containing at least one of these vertices to i in the corresponding graph C_U. So they are less than i under height order. By Algorithm 2, they are strictly lower than i. So i is outside of the range of all vertices that could collide with it.

Now we consider part E_L (the lower part,also the vertices of C_L): For any edge i in part E_L, for all the vertices that collide with i, there is a direct edge from those edges containing at least one of these vertices to i in the collision graph C_L. So they are less than i under height order. By Algorithm 3, these edges are strictly higher than i. So i is outside of the range of all vertices that could collide with it.

For any edge i in part E_U, if some vertex j in part E_L collides with i. There are two cases: j also shows up in E_U; j only shows up in E_L. For the second case, we know the range of vertex j is limited within part E_L, but edge i is in part E_U. By our algorithm, height of i is outside of the range of vertex j. For the first case, since all the edges that contain j in part E_U already are lower than edge i. It is obvious that all the edges that contain j in part E_L are also by height lower than i, so i is outside of the range of these vertices. As for the case when some vertex j in part E_U collides with any edge i in part E_L, the argument is analogous.

By now, we finish the analysis for all situations, and we have shown that Algorithm 2 and Algorithm 3 together provides us a collision-free height arrangement in the L-model for moving graph M. □

Proof (proof of theorem 1). Given two acyclic collision graphs C_L and C_U of M induced by E_U and E_L, respectively. Apply Algorithm 2 and Algorithm 3, we get a collision-free height arrangement for edges of M in its corresponding L-model. Theorem 1 follows from Proposition 2. □

We continue with Example 1, apply Algorithm 2 and 3 to it.

Example 3. We partition edges of M into two parts:

$$E_U = \{51, 52, 53, 54\}, E_L = \{61, 62, 63, 64, 71, 72, 73, 74\},$$

then from this two parts we construct the induced collision graphs C_U and C_L, respectively.

After applying Algorithm 2 and Algorithm 3, we get a collision-free height arrangement for a corresponding L-model as: $h(54) = 4, h(53) = 3, h(52) = 2, h(51) = 1, h(61) = 0, h(62) = -1, h(63) = -2, h(64) = -3,$ $h(71) = -4, h(72) = -5, h(73) = -6, h(74) = -7$.

M in this example is just one graph from the Dixon-1 moving graph family. In the next section we discuss this family of moving graphs.

Theorem 2. *The converse statement of Theorem 1 does not hold. That is to say, there exists a moving graph $M = (G, F)$ where $G = (V, E)$ that has a collision-free L-model but we cannot partition E into two parts such that the induced collision graph of M in both parts are acyclic.*

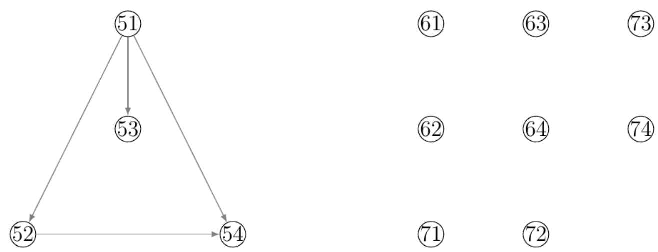

Fig. 3. Here on the left is the collision graph of M induced by $\{51, 52, 53, 54\}$ and on the right is the collision graph of M induced by $\{61, 62, 63, 64, 71, 72, 73, 74\}$.

Proof. We show a counter-example which we call S_2 moving graph [6]. Let $M = (G, F)$, where $G = (V, E)$, $V = \{v_i | 1 \le i \le 8, i \in \mathbb{N}\}$, $E = \{v_1v_2, v_1v_4, v_1v_5, v_8v_2, v_8v_4, v_8v_5, v_3v_2, v_3v_4, v_3v_5, v_1v_7, v_7v_6, v_5v_6, v_4v_6\}$.

$F = \{f_{v_i} : \mathbb{R} \to \mathbb{R}^2, t \mapsto (x_{v_i}(t), y_{v_i}(t)) | 1 \le i \le 8, i \in \mathbb{N}\}$.

In the following functions we have $a = 1$, $b = \frac{11}{5}$, $c = \frac{3}{2}$:

$$
\begin{aligned}
f_{v_1}(t) &= (-a\cos t - \sqrt{b^2 - a^2\sin^2 t}, -a\sin t - \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_2}(t) &= (a\cos t - \sqrt{b^2 - a^2\sin^2 t}, -a\sin t + \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_3}(t) &= (a\cos t + \sqrt{b^2 - a^2\sin^2 t}, a\sin t + \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_4}(t) &= (-a\cos t + \sqrt{b^2 - a^2\sin^2 t}, -a\sin t + \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_5}(t) &= (-a\cos t + \sqrt{b^2 - a^2\sin^2 t}, a\sin t - \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_6}(t) &= (-3a\cos t + \sqrt{b^2 - a^2\sin^2 t}, -a\sin t - \sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_7}(t) &= (-3a\cos t - \sqrt{b^2 - a^2\sin^2 t}, -a\sin t - 3\sqrt{c^2 - a^2\cos^2 t}),\\
f_{v_8}(t) &= (-a\cos t - \sqrt{b^2 - a^2\sin^2 t}, a\sin t + \sqrt{c^2 - a^2\cos^2 t}).
\end{aligned}
$$

After applying our collision-detection program, we get collision pairs in M as: $(v_6, v_1v_5), (v_3, v_1v_4), (v_8, v_1v_2), (v_8, v_2v_3), (v_4, v_2v_3), (v_4, v_3v_5), (v_6, v_3v_5), (v_3, v_4v_6),$ $(v_5, v_4v_6), (v_3, v_4v_8), (v_2, v_4v_8), (v_2, v_5v_8), (v_6, v_5v_8), (v_5, v_6v_7)$.

One can easily check that $h(v_6v_7) = 0, h(v_1v_4) = 1, h(v_4v_6) = 2, h(v_4v_8) = 3, h(v_3v_4) = 4, h(v_5v_6) = 5, h(v_5v_8) = 6, h(v_1v_5) = 7, h(v_1v_7) = 8, h(v_3v_5) = 9, h(v_2v_8) = 10, h(v_2v_3) = 11, h(v_1v_2) = 12$ is a collision-free height arrangement for an L-model of M.

However, from the collision information of M, we know that in its collision graph $C = (V_C, E_C)$, $\overrightarrow{e_1e_2}, \overrightarrow{e_2e_1}, \overrightarrow{e_1e_3}, \overrightarrow{e_3e_1}, \overrightarrow{e_2e_3}, \overrightarrow{e_3e_2} \in E_C$, where $e_1 = v_2v_3, e_2 = v_4v_6, e_3 = v_5v_8$. Then no matter how we try to bi-partition V_C, we will get two of e_1, e_2, e_3 in one partitioned group in which there is a multi-edge (cycle) in the induced collision graph.

So we cannot partition V_C into two parts such that the induced collision graph of both parts are acyclic. □

To sum up this section, how to decide whether a given moving graph has a collision-free L-model? We need to go through the following steps:

- 1) Collect collision information of the given moving graph M, namely the collision pairs in M.
- 2) Construct the collision graph of this moving graph, denote it as C.
- 3) Decide whether C fulfills partition condition. If yes, with our algorithm we can get a collision-free height arrangement for the corresponding L-model and we can construct it.

Remark 3. In the above steps, step 3) lacks an algorithm.

Definition 9 (multi-edged subgraph). *Let $C = (V, E)$ be a directed graph. Collect all vertices that are contained in at least one multi-edge in set S, substitute all multi-edges in $C[S]$ with a single non-directed edge. Then we get the* ***multi-edged subgraph*** *of C.*

Example 4. We give an example with Figure 2, denote the graph in it as C. The multi-edged subgraph C_1 of C is shown in Figure 4.

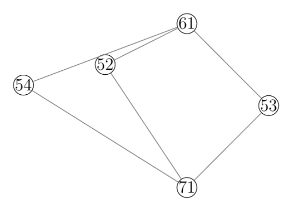

Fig. 4. Here is the multi-edged subgraph C_1 of graph C which is shown in Figure 2.

Now we give an algorithm for deciding whether C fulfills partition condition.

5 Dixon-1 graph

In this section we focus on Dixon-1 moving graph family.

Let m, n be positive integers. The graph is $K_{m,n}$, the complete undirected bipartite graph with $m+n$ vertices. Fix real numbers $0 < a_1 < ... < a_{m-1}$ and $0 < b_1 < ... < b_{n-1}$, $a_0 = b_0 = 0$. Two independent sets of vertices are $\{p_0, ..., p_{m-1}\}$ and $\{q_0, ..., q_{n-1}\}$. At time t, the coordinates of vertices of graph $K_{m,n}$ are:

$f_{p_0}(t) = (\sin t, 0)$,

$f_{p_i}(t) = (\pm\sqrt{a_i + \sin^2 t}, 0)$, for $i = 1, 2, ..., m-1$

$f_{q_0}(t) = (0, \cos t)$,

Algorithm 4: algorithm for deciding whether C fulfills partition condition

Input : a directed graph $C = (V_C, E_C)$

Output : If C fulfills partition condition, output the partition on V_C; otherwise, output "NO".

- Step 1: Construct the multi-edged subgraph of C, denote it as $U = (V_U E_U)$.
- Step 2: If U is not bipartite, return "NO"; otherwise, go to Step 3.
- Step 3: Find all bi-partitions for U, denote these bi-partitions as sequence $B_1, ..., B_m$, denote the B_i as (p_i, q_i). List all possible bi-partitions for vertices in $V_C \setminus V_U$ as $B'_1, ..., B'_k$, denote B'_i as (p'_i, q'_i). For i from 1 to m, for j from 1 to k, check whether $(p_i \cup p'_j, q_i \cup q'_j)$ is a partition of V_C such that the induced subgraph of both parts are acyclic. If any of them does fufill, return this partition and quit; otherwise, return "NO".

$f_{q_j}(t) = (0, \pm\sqrt{b_j + \cos^2 t})$, for $j = 1, 2, ..., n-1$.

For convenience, we represent distance between two vertices v_1, v_2 as $|v_1 v_2|$ instead of $\|f_{v_1}(t) - f_{v_2}(t)\|$. Also, we denote the first coordinate (x-coordinate) of a vertex v as $v|_x$ instead of $f_v(t)|_x$ and the second coordinate (y-coordinate) of it as $v|_y$ instead of $f_v(t)|_y$.

Theorem 3. *A Dixon-1 moving graph has and only has these four possible collision situations:*

- *1) p_0 collides with all edges containing vertex q_0 except for edge p_0q_0.*
- *2) $p_i (i > 0)$ collides with all edges $p_k q_0$, where $k > i$ and $p_k|_x \cdot p_i|_x > 0$.*
- *3) q_0 collides with all edges containing vertex p_0 except for edge p_0q_0.*
- *4) $q_i (i > 0)$ collides with all edges $q_k p_0$, where $k > i$ and $p_k|_y \cdot p_i|_y > 0$.*

Proof. $|p_0q_0| + |p_0p_i| = |p_iq_0| (i > 1)$ has solution in $\mathbb{R}$, so collision in 1) exists. If p_0 collides with $p_k q_l (k > 0, l > 0)$, then $q_l|_y = 0$. But $l > 0$, $|q_l|_y| = \sqrt{b_l + \cos^2 t} \neq 0 (b_l > 0)$. This is a contradiction. So p_0 does not collide with edges not containing q_0.

$|q_0p_i| + |p_ip_j| = |q_0p_j| (j > i, i > 0,\ p_j|_x \cdot p_i|_x > 0)$ has solution in $\mathbb{R}$, so collision in 2) exists. If $p_i (i > 0)$ collides with $p_k q_l (k > 0)$ and $p_k|_x \cdot p_i|_x < 0$, w.l.o.g., assume $p_i|_x > 0, p_k|_x < 0$. Then the collision leads to $p_k|_x \geq 0$. This is a contradiction! If $p_i (i > 0)$ collides with $p_k q_l (0 < k < i)$. Obviously we have $|q_l p_k| < |q_l p_i|$. Collision leads to $|q_l p_k| \geq |q_l p_i|$. This is a contradiction! If $p_i (i > 0)$ collides with those edges $p_k q_l (l > 0)$, $p_k|_x \cdot p_i|_x > 0$ $(k > i)$. Since $p_i|_y = p_k|_y = 0$, collision leads to $q_l|_y = 0 (l > 0)$. However, $|q_l|_y| = \sqrt{b_l + \cos^2 t} \neq 0$ $(b_l > 0)$. This is a contradiction. Hence $p_i (i > 0)$ only collides with all edges $p_k q_0$, where $k > i$ and $p_k|_x \cdot p_i|_x > 0$.

Because of the generalized symmetry of Dixon-1 moving graph, proof for 3) and 4) is analogous. □

Theorem 4. *Every Dixon-1 graph has a collision-free L-model.*

Proof. From the analysis above, we know that if for a given Dixon1 moving graph M, all the $p_i (1 \leq i \leq m-1)$ have the same sign in the $x-$coordinate and all the $q_j (1 \leq j \leq n-1)$ have the same sign in the $y-$coordinate. Then the set of collision pairs in M is strictly contained in the set of collision pairs in other Dixon-1 graphs that are isomorphic to it. W.l.o.g., we just need to analyze this situation.

First, I divide the edges into two groups as the following:
E_U: $\{q_0p_0, q_0p_1, ..., q_0p_{m-1}\}$
E_L: $\{q_1p_0, q_1p_1, ..., q_1p_{m-1}, q_2p_0, q_2p_1, ..., q_2p_{m-1}, ..., q_{n-1}p_0, q_{n-1}p_1, ..., q_{n-1}p_{m-1}\}$

In part E_U, in the induced collision graph C_U, there is a directed edge from q_0p_0 to all the other vertices. There is a directed edge from $q_0p_i(i > 0)$ to all $q_0p_k(k > i)$. Apparently there is no cycle in C_U. In part E_L, in the induced collision graph C_L, there is a directed edge from all $q_ip_l(i > 0$, l can take values from 0 to $m)$ to q_kp_0 if and only if $k > i$. From Theorem 3 we know that these are all the edges in the collision graph in part E_L. Since all edges are directed from some edge with lower q-vertex corner index to some edge with strictly higher q-vertex coner index, so there is no cycle in it.

Apply Algorithm 2 and Algorithm 3, we get: $h(q_0p_i) = i + 1$ and when $i > 0$, $h(q_ip_l) = -(i-1)(m+1) - l$.

From Theorem 1, Theorem 3, Algorithm 2, Algorithm 3 and our analysis above, we obtain that this height arrangement can realize a collision-free L-model for all Dixon-1 moving graphs. □

6 Dixon-2 graph

In this section we discuss another class of moving graphs, Dixon-2 moving graph [7].

The vertex-set is : $\{1, 2, 3, 4, 5, 6, 7, 8\}$. It is $K_{4,4}$ and the two independent sets of vertex-set are $\{1, 2, 3, 4\}, \{5, 6, 7, 8\}$. The coordinates of vertices are($a, b, c, d \in \mathbb{R}^+$, $c > d > a, b > a$):

$$f_1(t) = (\frac{a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, \frac{a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_2(t) = (-\frac{a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, \frac{a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_3(t) = (-\frac{a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, -\frac{a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_4(t) = (\frac{a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, -\frac{a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_5(t) = (\frac{-a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, \frac{-a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_6(t) = (-\frac{-a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, \frac{-a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_7(t) = (-\frac{-a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, -\frac{-a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$
$$f_8(t) = (\frac{-a\cdot\cos t+\sqrt{b^2-a^2\sin^2 t}}{2}, -\frac{-a\cdot\sin t+\sqrt{d^2-a^2\cos^2 t}}{2}),$$

If we denote the coordinates of 1 by (x_1, y_1) and the coordinates of 5 by (x_2, y_2), then x_1, y_1, x_2, y_2 fulfills the following system of equations, where $x_1, x_2, y_1, y_2 > 0$ always holds. The coordinates of other vertices are obtained by changing the sign of some of the coordinates in vertex 1 or 5.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$$
$$(x_1 + x_2)^2 + (y_1 - y_2)^2 = b^2$$
$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = c^2$$
$$(x_1 - x_2)^2 + (y_1 + y_2)^2 = d^2$$

Theorem 5. *For a Dixon-2 moving graph, collision only happens at these six moments:*

- *when vertex 1 collides with edge 52;*
- *when vertex 1 collides with edge 54;*
- *when vertex 1 collides with edge 53;*
- *when vertex 5 collides with edge 16;*
- *when vertex 5 collides with edge 18;*
- *when vertex 5 collides with edge 17.*

To sum up, vertex 1 collides with all edges containing vertex 5 except for edge 15; Vertex 5 collides with all edges containing vertex 1 except for edge 15. Situations for the other vertices are symmetrically analogous.

These are all the possible collisions for any Dixon-2 moving graph.

Proof. For convenience, we represent distance between two vertices x, y as $|xy|$ instead of $\|f_x(t) - f_y(t)\|$. When vertex 1 collides with edge 52, we have

$$|15| + |12| = |52| \iff$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + 2x_1 = \sqrt{(x_2 + x_1)^2 + (y_2 - y_1)^2}.$$

$$x_1 = \frac{b-a}{2}, x_2 = \frac{a+b}{2}, y_1 = y_2 = \frac{\sqrt{d^2 - a^2}}{2} \iff t = 2k\pi, \forall k \in \mathbb{Z}$$

Since we get real number solutions satisfying the defining equation system, so this collision happens. This collision situation is shown in the left picture in Figure 5. We solve similarly for the other five cases. As is shown in Figure 6.

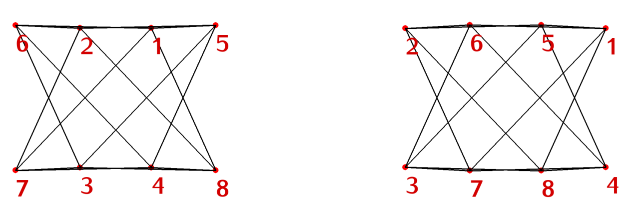

Fig. 5. 1→52(left), 5→16(right)

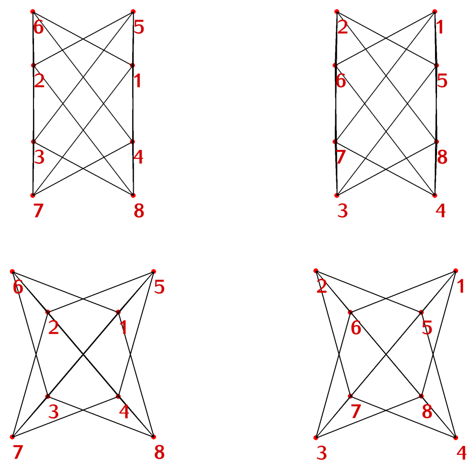

Fig. 6. 1→54(left top), 5→18(right top), 1→53(left bottom), 5→17(right bottom)

Take vertex 1 (or 5) for example, vertex 1 (or 5) actually collides with all edges containing vertex 5(or 1), except for edge 15. Since in our pre-condition, $x_1, y_1 > 0$ ($x_2, y_2 > 0$) always holds, it is impossible for vertex 1 (or 5) to collide with edges that does not show up in the first quadrant. From the definition we know that all edges containing vertex 5 (or 1) are exactly all edges that show up in the first quadrant. So vertex 1 (or 5) does not collide with any other edges. Because of the symmetry property of Dixon2 graph, proof for any other vertex is analogous. □

Lemma 1. *If there is collision-free L-model for a moving graph, then there exists a height arrangement of the model such that all edges have different height values.*

Proof. If there is a collision-free L-model of a moving graph, then we can always split those edges that are in the same layer so that we finally obtain a collision-free L-model of the graph where all edges have different height values. One can check the collision situation is not influenced by this split manipulation. □

Theorem 6. *There is no collision-free L-model for Dixon-2 moving graphs.*

Proof. Suppose there is a height arrangement that realizes a collision-free L-model for Dixon-2 moving graph. By Lemma 1, we can always assume all edges have different height values.

Denote height function as h. If $h(15) > h(1s) > h(5t)$, where $s \neq 5$ and $t \neq 1$, then edge 1s is within the range of vertex 5. Since vertex 5 collides with all edges containing 1 (except for 15), 5 also collides with edge 1s. This is a contradiction. If $h(15) < h(5s) > h(1t)$, where $s \neq 1$ and $t \neq 5$, then edge 5s is within the range of vertex 1. Since vertex 1 collides with all edges containing 5 (except for 15), 1 also collides with edge 5s. This is a contradiction. Thus, there cannot be any other edges containing vertex 1 lying in between edge 15 and any other edges containing vertex 5.

Also, there cannot be any other edges containing vertex 5 lying in between edge 15 and any other edges containing vertex 1. To conclude, range of vertex 1 and range of vertex 5 must only intersect on edge 15, i.e., 15 must be on the boundary of range of vertex 1 and range of vertex 5. W.l.o.g., assume edges containing 1 (except for edge 15) are all higher than the heights of edges containing 5 (except for edge 15). Then we obtain $h(1x) > h(1y) > h(1z) > h(15) > h(5u) > h(5v) > h(5w)$, where $\{x, y, z\} = \{6, 7, 8\}$, $\{u, v, w\} = \{2, 3, 4\}$.

Now I claim that $h(47) > h(53)$. Suppose this does not hold, i.e., $h(47) < h(53)$. Then we obtain $h(17) > h(53) > h(47)$. Now we see that edge 53 is within the range of vertex 7. Since vertex 7 collides with all edges containing 3 (except for edge 37), this is a contradiction. So we have $h(47) > h(53)$. Now I claim that $h(38) > h(54)$. Suppose this does not hold, i.e., $h(38) < h(54)$. Then we obtain $h(18) > h(54) > h(38)$. Now we see that edge 54 is within the range of vertex 8. Since vertex 8 collides with all edges containing 4 (except for edge 48), this is a contradiction. So we have $h(38) > h(54)$.

Then we try to arrange the heights of edge 47 and 38. If $h(47) > h(38)$, then we have $h(47) > h(38) > h(54)$. Now we see that edge 38 is within the range of vertex 4. Since vertex 4 collides with all edges containing 8 (except for edge 48), this is a contradiction. If $h(47) < h(38)$, then we have $h(53) < h(47) < h(38)$. Now we see that edge 47 is within the range of vertex 3. Since vertex 3 collides with all edges containing 7 (except for edge 37), this is a contradiction. Hence, no matter how we arrange edge 47 and edge 38, there must be some collision!

Hence, there is no zero-collision L-model realization for Dixon2 moving graph. □

Corollary 1. *Dixon-2 moving graph does not fulfill the condition in Theorem 1. That is to say, there is no partition of the edges of Dixon-2 moving graph into two parts E_L, E_U, such that the induced collision graphs C_L(by E_L) and C_U(by E_U) both are acyclic.*

Proof. Suppose not, then by Theorem 1 there is a height arrangement for the an L-model of Dixon-2 moving graph such that there is zero collision. However, this contradicts with Theorem 6. □

7 Acknowledgement

I thank Josef Schicho for posting this problem to me, the parametrization of Dixon-2 moving graph, and detailed modification suggestions etc. I thank Jan Legersky for reminding me of edge intersection case, the parametrization of S_2 moving graph, and detailed modification suggestions. I thank Ralf Hemmecke for helping in debugging the collision-detection program. I thank Jose capco for detailed modification suggestions.

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