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On C^2-finite sequences *

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Abstract

Holonomic sequences are widely studied as many objects interesting to mathematicians and computer scientists are in this class. In the univariate case, these are the sequences satisfying linear recurrences with polynomial coefficients and also referred to as D-finite sequences. A subclass are C-finite sequences satisfying a linear recurrence with constant coefficients.

We investigate the set of sequences which satisfy linear recurrence equations with coefficients that are C-finite sequences. These sequences are a natural generalization of holonomic sequences. In this paper, we show that C^2-finite sequences form a difference ring and provide methods to compute in this ring.

1 Introduction

Sequences that satisfy a linear recurrence with polynomial coefficients are known under the names holonomic, D-finite or P-recursive. If the recurrence coefficients are just constants, these sequences are also called C-finite or C-recursive. Many interesting combinatorial objects or coefficient sequences of special functions are of this type [4, 11]. In this paper, we define C^2-finite sequences as sequences satisfying a linear recurrence relation with C-finite coefficients. Holonomic and q-holonomic sequences are strictly contained in this set.

For holonomic functions or sequences, closure properties are a basic tool to systematically construct new holonomic objects from given ones and, more importantly, to automatically prove identities on holonomic objects. We set up C^2-finite sequences in a way that allows to derive and implement closure properties. The goal is to develop a toolkit for automated theorem proving as is already available for holonomic sequences and functions [10]. The main computational issue when working with this more general class compared to holonomic sequences is the presence of zero divisors.

To our knowledge, C^2-finite sequences have first been introduced formally by Kotek and Makowsky [13] in the context of graph polynomials. Thanatipanonda and Zhang [16] give an overview on different properties of polynomial, C-finite and holonomic sequences and consider the extension under the name X-recursive sequences. The setting in these articles is slightly different which leads to complications if one aims at developing an algorithmic approach.

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In this paper, we show that C^2-finite sequences form a difference ring with respect to termwise addition and termwise multiplication and present a first step towards setting up the theory of C^2-finite sequences algorithmically. An implementation in SageMath [17] is under development for proof-of-concept and later release. In Section 3, we provide the algebraic characterization of C^2-finite sequences that serves as the theoretical backbone, but cannot be used straightforward in a constructive way. Next, in Section 4, we consider in full detail the closure property addition of two C^2-finite sequences. The multiplication can be handled analogously. Finally, in Section 5, we state some of the classical closure properties such as partial sum or interlacing that can be derived similar to the case of holonomic sequences.

2 Preliminaries

In this section, we introduce some notation that is used throughout the paper. Let $\mathbb{K}$ be a computable field of characteristic zero and we denote by $\mathbb{K}^{\mathbb{N}}$ the set of sequences over $\mathbb{K}$. These sequences form a ring with termwise addition and multiplication (i.e., the Hadamard product). The shift operator

$$\sigma\colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad \sigma((a(n))_{n\in\mathbb{N}}) = (a(n+1))_{n\in\mathbb{N}}$$

is an endomorphism on $\mathbb{K}^{\mathbb{N}}$. A difference subring is a subring R of $\mathbb{K}^{\mathbb{N}}$ which is closed under shifts, i.e., σ is an endomorphism on R. The noncommutative ring of shift-operators over R is denoted by $R[\sigma]$ and elements $\mathcal{C} = c_0 + c_1\sigma + \cdots + c_r\sigma^r \in R[\sigma]$ act in the natural way as

$$\mathcal{C}a = (c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r))_{n\in\mathbb{N}}$$

on $a \in \mathbb{K}^{\mathbb{N}}$.

For a difference subring $R \subseteq \mathbb{K}^{\mathbb{N}}$ we denote by $R^\times \subseteq R$ the set of sequences which are units in $\mathbb{K}^{\mathbb{N}}$, i.e., sequences which are nonzero everywhere. This is a multiplicatively closed subset of R. Furthermore, $Q(R)$ denotes the localization of R with respect to $R^\times$. We can consider $Q(R)$ as a subring of $\mathbb{K}^{\mathbb{N}}$ by $((a/b)(n))_{n\in\mathbb{N}} = (a(n)/b(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ for $a/b \in Q(R)$.

The ring of C-finite sequences is a difference ring and we denote it by $\mathcal{R}_C$.

Definition 2.1. A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is called C^2*-finite* if there are sequences $c_0, \ldots, c_r \in \mathcal{R}_C$ with $c_r \in \mathcal{R}_C^\times$ such that

$$c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r) = 0,$$

for all $n \in \mathbb{N}$. We define the order $\operatorname{ord}(a)$ of a C^2-finite sequence a as the minimal such r.

Note that the set of C^2-finite sequences contains holonomic sequences (and as such C-finite sequences), since polynomial sequences are C-finite. A C^2-finite sequence is described completely by a finite amount of data: the recurrence coefficients $c_0, \ldots, c_r \in \mathcal{R}_C$ and initial values $a(0), \ldots, a(r-1)$. The recurrence coefficients in turn have a finite description of the same form. This way, C^2-finite sequences can be represented exactly on a computer.

In operator notation, a sequence $a \in \mathbb{K}^{\mathbb{N}}$ is C^2-finite if there is an $\mathcal{A} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \mathcal{R}_C^\times$ and $\mathcal{A}a = 0$. Note that for a sequence $c \in \mathcal{R}_C$ it is an open problem (the so called *Skolem-Problem* [14]) whether it can be decided algorithmically if $c \in \mathcal{R}_C^\times$. However, even if in practice it may not be possible to verify formally, usually it is easy to verify empirically.

Instead of working in the ring $\mathbb{K}^{\mathbb{N}}$ we could also work in the ring $\mathcal{S}_{\mathbb{K}} \coloneqq \mathbb{K}^{\mathbb{N}}/J$ for $J \coloneqq \bigcup_{i\in\mathbb{N}} \ker(\sigma^i)$ where two sequences are equal if they are equal from some term on (cf. [15]). This setting is also used in [13, 16]. Let us write $\pi\colon \mathbb{K}^{\mathbb{N}} \to \mathcal{S}_{\mathbb{K}}$ for the natural projection. We say that $a + J \in \mathcal{S}_{\mathbb{K}}$ is C^2-finite if there is an operator $\mathcal{A} \in \pi(\mathcal{R}_C)[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \pi(\mathcal{R}_C)^\times$ and

$\mathcal{A}(a+J) = 0 + J$. Equivalently, the sequence $a \in \mathbb{K}^{\mathbb{N}}$ satisfies a C^2-finite recurrence from some term $n_0 \in \mathbb{N}$ on. By shifting the recurrence by n_0 we would get a recurrence which holds for every $n \in \mathbb{N}$. The advantage of working over $\mathcal{S}_{\mathbb{K}}$ is that one can decide if an operator $\mathcal{A}$ is of the desired shape since it can be decided whether $\operatorname{lc}(\mathcal{A}) \in \pi(\mathcal{R}_C)^{\times}$ [3, 5]. For practical computations, one is still limited by the Skolem-Problem. Thus, and since in our setting we avoid certain technicalities, we stick to working over the ring $\mathbb{K}^{\mathbb{N}}$ as stated above.

We conclude this section by giving some first concrete examples of C^2-finite sequences. More examples can be found in [13] and [16].

Example 2.2. Let $(a(n))_{n\in\mathbb{N}}$ count the number of graphs on n labeled nodes (sequence A006125 in the OEIS [7]). Then, $a(n) = 2^{n(n-1)/2}$ and a is C^2-finite as

$$2^n a(n) - a(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Similarly, all sequences $(\alpha^{n^2})_{n\in\mathbb{N}}$ for $\alpha \in \mathbb{K}$ are C^2-finite. These grow faster than holonomic sequences [6, Proposition 1.2.1]. Hence, the set of C^2-finite sequences is a strict generalization of holonomic sequences.

Example 2.3. Let $(f(n))_{n\in\mathbb{N}}$ be the Fibonacci numbers. It was observed in [13] that

$$\begin{aligned} f(2n+3)(f(2n+1)f(2n+3) - f(2n+2)^2)f(n^2)& \\ +f(2n+2)(f(2n+3)+f(2n+1))f((n+1)^2)& \\ -f(2n+1)f((n+2)^2) &= 0 \end{aligned}$$

holds for all $n \in \mathbb{N}$. In particular, $(f(n^2))_{n\in\mathbb{N}}$ is C^2-finite (A054783 in the OEIS).

Example 2.4. Let f be as above, l denote the Lucas numbers (with $l(0) = 2, l(1) = 1$), and $\operatorname{Fib}(n,k) := \prod_{i=1}^{k} f(n-i+1)/f(k)$ the fibonomial coefficient. It has been shown [12, Theorem 1] that $\sum_{k=0}^{n} \operatorname{Fib}(2n+1,k) = \prod_{k=1}^{n} l(2k)$ for all $n \in \mathbb{N}$. This is an identity of C^2-finite sequences (A294349 in the OEIS).

3 Algebraic characterization

For a sequence $a \in \mathbb{K}^{\mathbb{N}}$ we consider the module of shifts

$$\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$$

over the ring $Q(\mathcal{R}_C)$ where the scalar multiplication is given by the Hadamard product of sequences in $\mathbb{K}^{\mathbb{N}}$. We now prove that this module is finitely generated if and only if the sequence is C^2-finite.

Theorem 3.1. *The following are equivalent:*

1. *The sequence a is C^2-finite.*
2. *There exists $\mathcal{A} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \mathcal{R}_C^{\times}$ and a C^2-finite sequence b with $\mathcal{A}a = b$.*
3. *The module $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$ over the ring $Q(\mathcal{R}_C)$ is finitely generated.*

Proof. (1) $\implies$ (3): Suppose $\mathcal{A} = c_0 + c_1\sigma + \cdots + c_r\sigma^r \in \mathcal{R}_C[\sigma]$ is an annihilator of a with $\operatorname{lc}(\mathcal{A}) = c_r \in \mathcal{R}_C^{\times}$, i.e., $\mathcal{A}a = 0$. Let $i \in \mathbb{N}$, then

$$\sigma^i \mathcal{A} = \sigma^i(c_0)\sigma^i + \cdots + \sigma^i(c_r)\sigma^{i+r}$$

and $\operatorname{lc}(\sigma^i \mathcal{A}) = \sigma^i(c_r) \in \mathcal{R}_C^\times$. Since $(\sigma^i \mathcal{A})a = \sigma^i(\mathcal{A}a) = 0$, we can write

$$\sigma^{i+r}(a) = -\frac{\sigma^i(c_0)}{\sigma^i(c_r)}\sigma^i(a) - \cdots - \frac{\sigma^i(c_{r-1})}{\sigma^i(c_r)}\sigma^{i+r-1}(a).$$

Hence, for all $i \in \mathbb{N}$ the sequence $\sigma^{i+r}a$ is a $Q(\mathcal{R}_C)$-linear combination of the sequences $\sigma^i a, \dots, \sigma^{i+r-1}a$. By induction, $\sigma^{i+r}a$ is a $Q(\mathcal{R}_C)$-linear combination of $a, \sigma a, \dots, \sigma^{r-1}a$, so the module $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(\mathcal{R}_C)}$ is generated by $a, \sigma a, \dots, \sigma^{r-1}a$.

(3) $\implies$ (1): As the module is finitely generated, we can write

$$\langle b_0, \dots, b_m \rangle_{Q(\mathcal{R}_C)} = \langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(\mathcal{R}_C)}$$

for some m and some sequences $b_0, \dots, b_m$. There exists an $r \in \mathbb{N}$ such that the elements b_j can be written as $b_j = \sum_{i=0}^{r-1} c_{i,j}\sigma^i a$ for some $c_{i,j} \in Q(\mathcal{R}_C)$. Then, $\sigma^r a$ is a $Q(\mathcal{R}_C)$-linear combination of $b_0, \dots, b_m$, so in particular a linear combination of the $a, \sigma a, \dots, \sigma^{r-1}a$. Hence, there exist sequences $c_0, \dots, c_{r-1} \in \mathcal{R}_C$ and $d_0, \dots, d_{r-1} \in \mathcal{R}_C^\times$ with

$$\sigma^r a = \frac{c_0}{d_0}a + \frac{c_1}{d_1}\sigma a + \cdots + \frac{c_{r-1}}{d_{r-1}}\sigma^{r-1}a.$$

Clearing denominators shows that a is C^2-finite of order at most r.

(2) $\implies$ (1): Since b is C^2-finite, there exists an operator $\mathcal{B} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{B}) \in \mathcal{R}_C^\times$ and $\mathcal{B}b = 0$. Then, $(\mathcal{B}\mathcal{A})a = \mathcal{B}(\mathcal{A}a) = \mathcal{B}b = 0$. Furthermore, $\operatorname{lc}(\mathcal{B}\mathcal{A}) \in \mathcal{R}_C^\times$.

(1) $\implies$ (2): We can choose the C^2-finite sequence $b = 0$. □

Analogous results like Theorem 3.1 for C-finite and holonomic sequences are often used to show that these sets form rings [11]. In these cases the base ring is a field and the key step makes use of the fact that submodules of finitely generated modules over fields (i.e., finite dimensional vector spaces) are again finitely generated. This holds more generally for Noetherian rings. However, the rings $\mathcal{R}_C$ and $Q(\mathcal{R}_C)$ are not Noetherian as the next example shows.

Example 3.2. Let $c_k \in \mathcal{R}_C$ with $c_k(n) - c_k(n+k) = 0$, for every $n \in \mathbb{N}$, and $c_k(0) = \cdots = c_k(k-2) = 1, c_k(k-1) = 0$ (i.e., c_k has a 0 at every k-th term and 1 else). Let $L_m \coloneqq \langle c_2, \dots, c_{2^m}\rangle$ be ideals in $\mathcal{R}_C$, then

$$L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots$$

is an infinitely properly ascending chain of ideals in $\mathcal{R}_C$. Therefore, $\mathcal{R}_C$ is not a Noetherian ring.

Hence, to use a similar argument for C^2-finite sequences, we construct a Noetherian subring $R \subseteq \mathcal{R}_C$ in the next theorem.

Theorem 3.3. *The set of C^2-finite sequences is a difference ring under termwise addition and termwise multiplication.*

Proof. Let a, b be C^2-finite sequences and $\mathcal{A} = c_0 + c_1\sigma + \cdots + c_{r_1}\sigma^{r_1}$ and $\mathcal{B} = d_0 + d_1\sigma + \cdots + d_{r_2}\sigma^{r_2}$ the corresponding annihilating operators with $c_0, \dots, c_{r_1}, d_0, \dots, d_{r_2} \in \mathcal{R}_C$.

For a C-finite sequence $c \in \mathcal{R}_C$ the $\mathbb{K}$-vector space $\langle \sigma^i c \mid i \in \mathbb{N}\rangle_{\mathbb{K}}$ is finitely generated. Then, also the $\mathbb{K}$-algebra

$$R_c \coloneqq \mathbb{K}[c, \sigma c, \sigma^2 c, \dots]$$

is finitely generated as an algebra. In particular, R_c is a Noetherian ring [1, Corollary 7.7].

Now, let $R \subsetneq \mathcal{R}_C$ be the smallest ring containing the Noetherian rings $R_{c_0}, \dots, R_{c_{r_1}}$, $R_{d_0}, \dots, R_{d_{r_2}}$. In particular, this ring R is finitely generated as a ring and therefore, R and

$Q(R)$ are Noetherian rings [1, Corollary 7.7, Proposition 7.3]. From the proof of Theorem 3.1 it is clear that $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(R)}$ and $\langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(R)}$ are both finitely generated $Q(R)$-modules. Hence, also the modules

$$\langle \sigma^i (a+b) \mid i \in \mathbb{N} \rangle_{Q(R)} \subseteq \langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(R)} + \langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(R)}$$

and

$$\langle \sigma^i (ab) \mid i \in \mathbb{N} \rangle_{Q(R)} \subseteq \langle \sigma^i (a) \sigma^j (b) \mid i, j \in \mathbb{N} \rangle_{Q(R)}$$

are finitely generated as they are submodules of finitely generated modules over a Noetherian ring. Again, from the proof of Theorem 3.1 we can see that $a+b$ and ab are C^2-finite. Therefore, the set of C^2-finite sequences is a ring.

The operator

$$\tilde{\mathcal{A}} := \sigma(c_0) + \sigma(c_1)\sigma + \cdots + \sigma(c_{r_1})\sigma^{r_1} \in \mathcal{R}_C[\sigma]$$

annihilates σa as

$$\tilde{\mathcal{A}}(\sigma a) = (\tilde{\mathcal{A}}\sigma) a = (\sigma \mathcal{A}) a = \sigma(\mathcal{A} a) = 0.$$

Furthermore, we have $\operatorname{lc}(\tilde{\mathcal{A}}) = \sigma(c_{r_1}) \in \mathcal{R}_C^\times$. Hence, the ring of C^2-finite sequences is also closed under shifts. □

In [13, Theorem 1] it was shown that certain sparse subsequences of C-finite sequences are C^2-finite. Example 2.3 given earlier is just a special case of this. We provide an easier proof for a similar result which uses the closed-form representation of C-finite sequences.

Corollary 3.4. *Let c be a C-finite sequence over the field $\mathbb{K}$ and $j, k, l \in \mathbb{N}$. Then,*

$$(c(jn^2 + kn + l))_{n \in \mathbb{N}}$$

is C^2-finite over the splitting field $\mathbb{L}$ of the characteristic polynomial of c.

Proof. By the closed-form representation of C-finite sequences (cf. [11, Theorem 4.1]) c is an $\mathbb{L}$-linear combination of sequences d with $d(n) = n^i \alpha^n$ for $i \in \mathbb{N}$ and $\alpha \in \mathbb{L}$. Then,

$$d(jn^2 + kn + l) = (jn^2 + kn + l)^i (\alpha^j)^{n^2} (\alpha^k)^n \alpha^l.$$

Therefore, the sequence $(d(jn^2 + kn + l))_{n \in \mathbb{N}}$ is C^2-finite as it is the product of C-finite sequences and the C^2-finite sequence $((\alpha^j)^{n^2})_{n \in \mathbb{N}}$ over $\mathbb{L}$. Since C^2-finite sequences are also closed under $\mathbb{L}$-linear combinations, $(c(jn^2 + kn + l))_{n \in \mathbb{N}}$ is C^2-finite. □

4 Closure property addition

Classically, closure properties for holonomic functions or sequences are computed using an ansatz method [10]. We describe such an approach for the addition of two C^2-finite sequences. The same technique can also be used for the multiplication of two sequences.

Let a, b be C^2-finite. Then, we have recurrences

$$\begin{aligned} \tilde{c}_0(n) a(n) + \cdots + \tilde{c}_{r_1 - 1}(n) a(n + r_1 - 1) + \tilde{c}_{r_1}(n) a(n + r_1) &= 0, \\ \tilde{d}_0(n) b(n) + \cdots + \tilde{d}_{r_2 - 1}(n) b(n + r_2 - 1) + \tilde{d}_{r_2}(n) b(n + r_2) &= 0, \end{aligned}$$

for all $n \in \mathbb{N}$, for $\tilde{c}_0, \dots, \tilde{c}_{r_1-1}, \tilde{d}_0, \dots, \tilde{d}_{r_2-1} \in \mathcal{R}_C$ with leading coefficients $c_{r_1}, d_{r_2} \in \mathcal{R}_C^\times$. Therefore,

$$c_0(n)a(n) + \cdots + c_{r_1-1}(n)a(n+r_1-1) + a(n+r_1) = 0,$$
$$d_0(n)b(n) + \cdots + d_{r_2-1}(n)b(n+r_2-1) + b(n+r_2) = 0,$$

for all $n \in \mathbb{N}$, with $c_0, \dots, c_{r_1-1}, d_0, \dots, d_{r_2-1} \in Q(\mathcal{R}_C)$. To get a recurrence for $a+b$ we make an ansatz of some order s with unknown coefficients $x_0, \dots, x_{s-1} \in Q(\mathcal{R}_C)$:

$$x_0(n)(a(n)+b(n)) + \cdots + x_{s-1}(n)(a(n+s-1)+b(n+s-1)) + (a(n+s)+b(n+s)) = 0.$$

Repeated application of the recurrences and collecting $a(n+i)$ for $i = 0, \dots, r_1-1$ and $b(n+i)$ for $i = 0, \dots, r_2-1$ yields

$$\sum_{i=0}^{r_1-1} \left(\alpha_i(n) + \sum_{j=0}^{s-1} \alpha_{i,j}(n)x_j(n) \right) a(n+i) + \sum_{i=0}^{r_2-1} \left(\beta_i(n) + \sum_{j=0}^{s-1} \beta_{i,j}(n)x_j(n) \right) b(n+i) = 0$$

for some $\alpha_i, \alpha_{i,j}, \beta_i, \beta_{i,j} \in Q(\mathcal{R}_C)$. This equation is certainly true for all n if the coefficient sequences of $a(n+i)$ and $b(n+i)$ are zero. This yields a linear inhomogeneous system. To write it concisely, let us denote

$$u_j^\top = (\alpha_{0,j}, \dots, \alpha_{r_1-1,j}), \quad v_j^\top = (\beta_{0,j}, \dots, \beta_{r_2-1,j}),$$

for all $j = 0, \dots, s-1$,

$$u_s^\top = (\alpha_0, \dots, \alpha_{r_1-1}), \quad v_s^\top = (\beta_0, \dots, \beta_{r_2-1})$$

and

$$w_j^\top = (u_j, v_j) \in Q(\mathcal{R}_C)^{r_1+r_2},$$

for $j = 0, \dots, s$, and $x^\top = (x_0, \dots, x_{s-1}) \in Q(\mathcal{R}_C)^s$, as well as the matrices $U = (u_0, \dots, u_{s-1})$ and $V = (v_0, \dots, v_{s-1})$. Now, the system that we obtain from equating the coefficient sequences to zero reads as

$$\begin{pmatrix} U \\ V \end{pmatrix} x = -w_s.$$

In the next section, we show how the vectors w_j can be computed. Then, in Section 4.2, we see that the order of the ansatz s can be chosen big enough such that the inhomogeneous system has a solution in $\mathbb{K}^s$ at every term. Finally, from Lemma 4.5 it follows that there is a solution $x \in Q(\mathcal{R}_C)^s$ of the inhomogeneous system.

In the case that one of the C^2-finite sequences has order 1, the inhomogeneous system has a special structure. We use this to derive a bound for the order of the addition of the two sequences in Section 4.3.

4.1 Computing the ansatz

Let a be C^2-finite of order r with recurrence

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0,$$

for all $n \in \mathbb{N}$, and for $c_0, \dots, c_{r-1} \in Q(\mathcal{R}_C)$. We write the components of a vector $u_j \in Q(\mathcal{R}_C)^r$ as $u_{j,i}$ for $i = 0, \dots, r-1$. The j-th unit vector is denoted by $e_j \in Q(\mathcal{R}_C)^r$ for $j = 0, \dots, r-1$. Note that, e.g., $e_0(n) = (1, 0, \dots, 0)$, for all $n \in \mathbb{N}$.

The following lemma shows a straightforward recurrence which can be used to compute the vectors u_j in the ansatz matrix.

Lemma 4.1. *Let $u_j \coloneqq e_j \in Q(\mathcal{R}_C)^r$ be the j-th unit vector for $j = 0, \dots, r-1$. Now, define*

$$u_j(n) \coloneqq -\sum_{k=0}^{r-1} c_k(n+j-r)u_{j-r+k}(n), \qquad \text{for all } n \in \mathbb{N}, \tag{1}$$

inductively. These $u_j(n)$ satisfy

$$a(n+j) = \sum_{i=0}^{r-1} u_{j,i}(n)a(n+i), \qquad \text{for all } n \in \mathbb{N}, \tag{2}$$

for all $j \in \mathbb{N}$.

Proof. Shifting the defining recurrence of $a(n)$ yields

$$a(n+j) = -\sum_{k=0}^{r-1} c_k(n+j-r)a(n+j-r+k), \qquad \text{for all } n \in \mathbb{N},$$

for $j \geq r$. We show equation (2) by induction on j. It clearly holds for $j = 0, \dots, r-1$ by the definition of the u_j. Let $n \in \mathbb{N}$ and let us assume that equation (2) holds for $a(n), \dots, a(n+j-1)$. Then,

$$\begin{aligned} \sum_{i=0}^{r-1} u_{j,i}(n)a(n+i) &= -\sum_{i=0}^{r-1}\sum_{k=0}^{r-1} c_k(n+j-r)u_{j-r+k,i}(n)a(n+i) \\ &= -\sum_{k=0}^{r-1} c_k(n+j-r)a(n+j-r+k) = a(n+j). \end{aligned}$$

□

A different way to compute the vectors u_j is to use the companion matrix of a sequence. The companion matrix M_a of the sequence a is defined as

$$M_a \coloneqq \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{r-1} \end{pmatrix} \in Q(\mathcal{R}_C)^{r \times r}.$$

Lemma 4.2. *Let M_a be the companion matrix of a. Let*

$$u_0 \coloneqq e_0 = (1, 0, \dots, 0)$$

and define

$$u_j(n) \coloneqq M_a(n)u_{j-1}(n+1), \qquad \text{for all } n \in \mathbb{N},$$

inductively for $j \geq 1$.

1. *These u_j are identical to the vectors from Lemma 4.1.*
2. *The u_j satisfy equation (2).*

Proof. (1): Clearly $u_j = e_j$ for $j = 0, \dots, r-1$ by the definition of the companion matrix. For $j \geq r$ we show that equation (1) from Lemma 4.1 is satisfied using induction on j. For $j = r$ we have

$$u_r(n) = (-c_0(n), \dots, -c_r(n)), \quad \text{for all } n \in \mathbb{N},$$

by the definition of the companion matrix and therefore,

$$-\sum_{k=0}^{r-1} c_k(n) u_k(n) = -\sum_{k=0}^{r-1} c_k(n) e_k(n) = u_r(n), \quad \text{for all } n \in \mathbb{N}.$$

Now, we assume that equation (1) from Lemma 4.1 holds for $j-1$, i.e.,

$$u_{j-1}(n) = -\sum_{k=0}^{r-1} c_k(n+j-1-r) u_{j-1-r+k}(n), \tag{3}$$

for all $n \in \mathbb{N}$. Using equation (3) shifted $n \to n+1$ and the definition of the u_j we have

$$\begin{aligned} -\sum_{k=0}^{r-1} c_k(n+j-r) u_{j-r+k}(n) &= -M_a(n) \sum_{k=0}^{r-1} c_k(n+j-r) u_{j-1-r+k}(n+1) \\ &= M_a(n) u_{j-1}(n+1) = u_j(n), \end{aligned}$$

for all $n \in \mathbb{N}$.

(2): Follows directly from part (1) and Lemma 4.1. ☐

Consider two C^2-finite sequences a, b. To compute the vectors w_j in the ansatz matrix for $a+b$ we can concatenate the vectors which we get from Lemma 4.1. Alternatively, following the approach from [8], we can use Lemma 4.2 and compute $w_j(n) = M(n) w_{j-1}(n+1)$, for $n \in \mathbb{N}$, where

$$M = M_a \oplus M_b = \begin{pmatrix} M_a & 0 \\ 0 & M_b \end{pmatrix}$$

is the direct sum of the companion matrices of a and b.

For the product ab we can do an analogous method and use the Kronecker product

$$M = M_a \otimes M_b$$

of the matrices M_a and M_b.

4.2 Computations in the ring

In Section 4.1 we have seen constructive ways to compute the ansatz matrix. Lemma 4.3 below yields that this ansatz can be chosen large enough such that the corresponding inhomogeneous system has a solution at every term. Lemma 4.5 then states that such termwise solvable systems are even solvable in the C-finite sequence ring. For both results we adapt some techniques which were used in [13]. As a consequence, we obtain a (semi-)constructive way to compute the addition and multiplication in the C^2-finite sequence ring.

Lemma 4.3. *Let a, b be C^2-finite sequences. Then, the order s of the ansatz for the sum $a+b$ and the product ab can be chosen in such a way that the corresponding inhomogeneous system has a solution at every term.*

Proof. Let $w_0, w_1, \ldots \in Q(\mathcal{R}_C)^r$ (with $r = r_1 + r_2$ if we consider the sum $a+b$ and $r = r_1 r_2$ if we consider the product ab) be the columns of the ansatz matrix. Let $S \subsetneq \mathcal{R}_C$ be the smallest ring containing all $\mathbb{K}$-algebras $\mathbb{K}[c, \sigma c, \sigma^2 c, \ldots]$ where c is a coefficient in the annihilating operator of a or b. In the proof of Theorem 3.3 we have seen that $Q(S)$ is a Noetherian ring. Now, denote $A_j := (w_0, \ldots, w_j) \in Q(S)^{r\times(j+1)}$. Furthermore, let $I_j^{(t)} \trianglelefteq Q(S)$ be the ideals generated by the minors of order t of A_j. For fixed $t \in \{0, \ldots, r\}$, these $I_j^{(t)}$ form an increasing chain of ideals. Let $s \in \mathbb{N}$ be such that $I_{s-1}^{(t)} = I_s^{(t)}$ for all $t \in \{0, \ldots, r\}$. Then, $A_{s-1}(n)x(n) = -w_s(n)$ has a solution for every n: Suppose

$$t := \operatorname{rank}(A_s(n)) > \operatorname{rank}(A_{s-1}(n)).$$

Then, there exists a nonzero minor $\phi(n)$ of order t of $A_s(n)$. On the other hand, all minors $\phi_0(n) = \cdots = \phi_m(n) = 0$ of order t of $A_{s-1}(n)$ are zero. By the choice of s, the nonzero minor $\phi(n)$ is a $Q(S)$-linear combination of the minors $\phi_0(n), \ldots, \phi_m(n)$, a contradiction. Hence, $A_{s-1}(n)$ and $A_s(n)$ have equal rank and, by the Rouché-Capelli theorem, the linear equation has a solution. $\square$

The proof of Lemma 4.3 is not constructive as the Noetherian ring only gives us the existence of the number s. To make this argument constructive we would need to be able to solve instances of the ideal membership problem over $Q(\mathcal{R}_C)$.

The order of the addition and multiplication of C-finite sequences is bounded by the sum and product of the orders of the sequences respectively. Lemma 4.3 does not provide such bounds. The next example shows that these classical bounds do not work in some cases.

Example 4.4. Consider

$$(-1)^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Actually, a is also C-finite of order 2. Making an ansatz of order 2 for the sequence $a+b$ with coefficients $x_0, x_1 \in Q(\mathcal{R}_C)$ yields the linear system

$$\begin{pmatrix} 1 & -(-1)^n \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0(n) \\ x_1(n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This system is not solvable for even $n \in \mathbb{N}$. Hence, our technique cannot yield a recurrence for $a+b$ of order 2. However, with an ansatz of order 3 we get the recurrence

$$\begin{aligned} &\left(\tfrac{1}{2}(-1)^{n+1} + \tfrac{1}{2}\right)(a(n) + b(n)) \\ &+ \left(\tfrac{1}{2}(-1)^n + \tfrac{1}{2}\right)(a(n+2) + b(n+2)) \\ &\qquad\qquad + (a(n+3) + b(n+3)) = 0, \end{aligned}$$

for every $n \in \mathbb{N}$. Setting up a classical homogeneous ansatz as in [16] yields the recurrence

$$((-1)^n + 1)(a(n) + b(n)) + 2(a(n+1) + b(n+1)) + (1 - (-1)^n)(a(n+2) + b(n+2)) = 0$$

with a leading coefficient which has infinitely many zeros. Such a recurrence fits in the framework of X-recursive sequences from [16] but is not a C^2-finite recurrence in our sense.

In order to show how to solve systems over the ring $Q(\mathcal{R}_C)$ we use the Skolem-Mahler-Lech Theorem [5]. It states that the zeros of a sequence $c \in \mathcal{R}_C$ (and therefore also $c \in Q(\mathcal{R}_C)$)

are exactly at finitely many arithmetic progressions from some term on. Hence, for a sequence $c \in \mathcal{R}_C$ there exist $n_0, p \in \mathbb{N}$ such that

$$(c(n_0 + pk), \ldots, c(n_0 + pk + p - 1))$$

has the same zero-pattern for every $k \in \mathbb{N}$. This number p is called the zero-cycle period of the sequence c.

Lemma 4.5. *Let $A \in Q(\mathcal{R}_C)^{r \times s}$ and $w \in Q(\mathcal{R}_C)^r$. Suppose the system $A(n)x(n) = w(n)$ has a solution for every $n \in \mathbb{N}$. Then, there is a solution $x \in Q(\mathcal{R}_C)^s$ such that $Ax = w$ in $Q(\mathcal{R}_C)$.*

Proof. All minors of A are sequences in $Q(\mathcal{R}_C)$. Consider the set of all these. By the Skolem-Mahler-Lech Theorem the zeros of these minors are cyclic. Let $p \in \mathbb{N}$ be the common zero-cycle period of all minors from some term $n_0 \in \mathbb{N}$ on.

We write $A = (w_0, \ldots, w_{s-1})$ for $w_0, \ldots, w_{s-1} \in Q(\mathcal{R}_C)^r$. Now, for every $m \in \{n_0, \ldots, n_0 + p - 1\}$ we can choose a subset $j_m \subseteq \{0, \ldots, s-1\}$ such that the vectors $\{w_j(m) \mid j \in j_m\} \subseteq \mathbb{K}^r$ are maximally linearly independent, i.e., they are linearly independent and generate the same subspace as $\{w_0(m), \ldots, w_{s-1}(m)\}$. By the choice of n_0 and p this is also true for all $n = m + pk$ for $k \in \mathbb{N}$, i.e., the vectors $\{w_j(m + pk) \mid j \in j_m\} \subseteq \mathbb{K}^r$ are maximally linearly independent for every $k \in \mathbb{N}$. Let us denote by $A_m \in Q(\mathcal{R}_C)^{r \times |j_m|}$ the submatrix of A where we keep the columns w_j with $j \in j_m$.

For every m we can solve the system

$$A_m(m + pk)x_m(k) = w(m + pk), \qquad \text{for all } k \in \mathbb{N}, \tag{4}$$

using the Moore-Penrose-Inverse: By the choice of m, p, n_0, the matrix $A_m(m + pk)$ has linear independent columns for every $k \in \mathbb{N}$. Therefore, the Gramian matrix

$$G(k) = A_m^T(m + pk)A_m(m + pk)$$

is regular for every k and $(\det(G(k)))_{k \in \mathbb{N}} \in \mathcal{R}_C^\times$. Now, let

$$x_m(k) = \frac{1}{\det(G(k))} \operatorname{cof}\left(G(k)\right) A_m^T(m + pk)w(m + pk)$$

where $\operatorname{cof}(\cdot)$ denotes the transposed cofactor matrix. Then, since equation (4) has a termwise solution, $(x_m(k))_{k \in \mathbb{N}} \in Q(\mathcal{R}_C)^{|j_m|}$ satisfies equation (4) by the theory of Moore-Penrose-Inverses. Let $x'_m \in Q(\mathcal{R}_C)^s$ be the vector where we add $0 \in Q(\mathcal{R}_C)$ at the indices $j \in j_m - \{0, \ldots, s-1\}$.

Now, the solution x for the entire system can be computed as the interlacing of $x'_{n_0}, \ldots, x'_{n_0+p-1}$ from n_0 on and the first n_0 values can be computed explicitly. Then, $x \in Q(\mathcal{R}_C)^s$ as $Q(\mathcal{R}_C)$ is closed under interlacing and specifying finitely many initial values. ☐

The arithmetic progressions from the Skolem-Mahler-Lech Theorem can be found effectively. Hence, the zero-cycle period of a C-finite sequence can be computed. It is, however, not known whether the index $n_0 \in \mathbb{N}$ such that the zeros beyond this index are cyclic can be found algorithmically (cf. Skolem-Problem [14]). Hence, the proof of Lemma 4.5 is not constructive in general. However, in many cases this index n_0 can be computed (or at least estimated empirically). If we can compute the zeros of the minors of the matrix A, then the proof of Lemma 4.5 gives an algorithm to compute a solution.

Lemma 4.5 also shows a possible algorithm to solve the ideal membership problem in $Q(\mathcal{R}_C)$ from Lemma 4.3: The problem whether $c \in \langle d_1, \ldots, d_s \rangle$ for $c, d_1, \ldots, d_s \in Q(\mathcal{R}_C)$ is equivalent to solving the inhomogeneous equation $dx = c$ with $d = (d_1, \ldots, d_s)$ for unknown $x \in Q(\mathcal{R}_C)^s$. With

Lemma 4.5 we can compute a possible solution x. Then, we can check with closure properties whether $dx = c$ indeed holds in $Q(\mathcal{R}_C)$. If it does, we have shown that c is in the ideal. If $dx \neq c$, then c is not in the ideal, because otherwise we would have a termwise solution and therefore with Lemma 4.5 a solution in $Q(\mathcal{R}_C)$.

Lemma 14 in [13] states that the components $w_j \in \mathbb{K}^{\mathbb{N}}$ of a vector $w = (w_0, \dots, w_{r-1})$ are C^2-finite if the vector satisfies a recurrence of the form $w(n+1) = M(n)w(n)$, for every $n \in \mathbb{N}$, for $M \in \mathcal{R}_C^{r\times r}$. Since the existence of the sequence s_n in the proof of Lemma 14 is not guaranteed, their approach seems to not work for certain examples. We use the same idea from Lemma 14 to set up an inhomogeneous linear system over $\mathcal{R}_C$. Lemma 4.5, which shows how to solve such inhomogeneous systems over $\mathcal{R}_C$, is also based on the proof of Lemma 14. The difference in our approach is that we do not assume that the inhomogeneous system has a certain fixed size (which is determined by $\operatorname{ord}(a), \operatorname{ord}(b)$) but that the size of this system also depends on the C-finite coefficients in the recurrences. Example 4.4 shows that this flexible approach is sometimes really needed.

4.3 Bounds for addition

The order of adding and multiplying C-finite sequences is bounded by the sum and product of the orders of the sequences respectively. Example 4.4 shows that this cannot be achieved in our approach for C^2-finite sequences. However, we can show some bounds in the special case where we add two C^2-finite sequences where one of the sequences has order 1.

In this section we assume that a is C^2-finite of order r and b is C^2-finite of order 1 satisfying the recurrences

$$\begin{aligned} c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0, \\ d(n)b(n) + b(n+1) = 0, \end{aligned}$$

for all $n \in \mathbb{N}$, with $c_0, \dots, c_{r-1}, d \in Q(\mathcal{R}_C)$.

Let u_j, v_j be the coefficients for the iterated recurrence of a and b as defined in Lemma 4.1, respectively. Note that $v_0 = 1$ and

$$v_j(n) = v_{j,0}(n) = -d(n+j-1)v_{j-1}(n), \qquad \text{for all } n \in \mathbb{N},$$

for $j \geq 1$. Therefore,

$$v_j(n) = (-1)^j d(n)d(n+1)\cdots d(n+j-1) \tag{5}$$

for all $j, n \in \mathbb{N}$.

As in the proof of Lemma 4.3 let

$$\phi_j \coloneqq \det(w_0, \dots, w_{r-1}, w_j) \in Q(\mathcal{R}_C)$$

for $j \geq 0$ with $w_j = (u_j, v_j)$. Let $v = (v_0, \dots, v_{r-1}) \in Q(\mathcal{R}_C)^r$ and let $I \in \mathbb{K}^{r\times r}$ be the identity matrix. Then,

$$\phi_j(n) = \begin{vmatrix} I & u_j(n) \\ v(n) & v_j(n) \end{vmatrix} = v_j(n) - \sum_{i=0}^{r-1} u_{j,i}(n)v_i(n),$$

for all $n \in \mathbb{N}$.

For $j < r$ we have $\phi_j = 0$ as the matrix has linear dependent columns. For $j = r$ we have $u_r = -(c_0, \ldots, c_{r-1})$ and therefore,

$$\phi_r(n) = v_r(n) + \sum_{i=0}^{r-1} c_i(n) v_i(n), \tag{6}$$

for every $n \in \mathbb{N}$.

Lemma 4.6. *For $j \geq r$ we have*

$$\phi_j(n) = -\sum_{i=0}^{r-1} c_i(n+j-r)\phi_{j-r+i}(n) + v_{j-r}(n)\phi_r(n+j-r),$$

for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. With the definition of the w_j we have

$$\begin{aligned}
\phi_j(n) &= \det(w_0(n), \ldots, w_{r-1}(n), w_j(n)) \\
&= -\sum_{i=0}^{r-2} \underbrace{\begin{vmatrix} I & c_i(n+j-r)u_{j-r+i}(n) \\ v(n) & 0 \end{vmatrix}}_{=:x_{j,i}(n)} - \underbrace{\begin{vmatrix} I & c_{r-1}(n+j-r)u_{j-1}(n) \\ v(n) & d(n+j-1)v_{j-1}(n) \end{vmatrix}}_{=:y_j(n)}.
\end{aligned}$$

First, we compute $x_{j,i}(n)$ and $y_j(n)$:

$$\begin{aligned}
x_{j,i}(n) &= \begin{vmatrix} I & c_i(n+j-r)u_{j-r+i}(n) \\ v(n) & 0 \end{vmatrix} = -c_i(n+j-r)\sum_{k=0}^{r-1} u_{j-r+i,k}(n)v_k(n) \\
&= c_i(n+j-r)\phi_{j-r+i}(n) - c_i(n+j-r)v_{j-r+i}(n)
\end{aligned}$$

and

$$\begin{aligned}
y_j(n) &= \begin{vmatrix} I & c_{r-1}(n+j-r)u_{j-1}(n) \\ v(n) & d(n+j-1)v_{j-1}(n) \end{vmatrix} = -v_j(n) - c_{r-1}(n+j-r)\sum_{i=0}^{r-1} u_{j-1,i}(n)v_i(n) \\
&= -v_j(n) + c_{r-1}(n+j-r)\phi_{j-1}(n) - c_{r-1}(n+j-r)v_{j-1}(n).
\end{aligned}$$

Combining both yields

$$\begin{aligned}
\phi_j(n) &= -\sum_{i=0}^{r-2} c_i(n+j-r)\phi_{j-r+i}(n) + \sum_{i=0}^{r-2} c_i(n+j-r)v_{j-r+i}(n) \\
&\quad + v_j(n) - c_{r-1}(n+j-r)\phi_{j-1}(n) + c_{r-1}(n+j-r)v_{j-1}(n) \\
&= -\sum_{i=0}^{r-1} c_i(n+j-r)\phi_{j-r+i}(n) + \sum_{i=0}^{r-1} c_i(n+j-r)v_{j-r+i}(n) + v_j(n).
\end{aligned}$$

Because of equation (5) we have $v_{j-r+i}(n) = v_i(n+j-r)v_{j-r}(n)$ and $v_j(n) = v_r(n+j-r)v_{j-r}(n)$. Hence, with equation (6), we have

$$\phi_j(n) = -\sum_{i=0}^{r-1} c_i(n+j-r)\phi_{j-r+i}(n) + v_{j-r}(n)\phi_r(n+j-r).$$

$\square$

Lemma 4.7. *Let a, b be C^2-finite of order r and 1 respectively. Then, $a + b$ has order at most $n_0 + r$ if there exists an $n_0 \in \mathbb{N}$ with $\phi_r(n) = 0$ for $n \geq n_0$. Otherwise, $a + b$ has order at most $\operatorname{ord}(\phi_r) + r$.*

Proof. Let

$$\begin{aligned} c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) &= 0, \\ d(n)b(n) + b(n+1) &= 0, \end{aligned}$$

for all $n \in \mathbb{N}$, with $c_0, \ldots, c_{r-1}, d \in Q(\mathcal{R}_C)$.

If $\phi_r(n) = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, we can shift the sequences by n_0, choose r for the order of the ansatz of $a + b$ and the corresponding system has a solution for every $n \in \mathbb{N}$. Now, shifting the recurrence by n_0 we can specify the initial values $(a + b)(n)$ for $n < n_0$.

Otherwise, we choose the order of the ansatz as $s \coloneqq \operatorname{ord}(\phi_r) + r$. We show that the corresponding linear system has a solution for every $n \in \mathbb{N}$: If one of the $\phi_r(n), \ldots, \phi_{s-1}(n)$ is nonzero, the system has a solution as we have r linearly independent vectors in $\mathbb{K}^r$. Now, assume that $\phi_r(n) = \cdots = \phi_{s-1}(n) = 0$. By the choice of s, the set $\{\phi_r(n), \ldots, \phi_r(n + s - r - 1)\}$ contains a nonzero element $\phi_r(n + s_n - r) \neq 0$ for some $s_n \in \{r, \ldots, s - 1\}$. Then, $\phi_{s_n}(n) = 0$ and by Lemma 4.6

$$\begin{aligned} \phi_{s_n}(n) &= -\sum_{i=0}^{r-1} c_i(n + s_n - r)\phi_{s_n - r + i}(n) + v_{s_n - r}(n)\phi_r(n + s_n - r) \\ &= -\sum_{i=0}^{r-1} c_i(n + s_n - r)0 + v_{s_n - r}(n)\phi_r(n + s_n - r) = 0. \end{aligned}$$

Therefore, $v_{s_n - r}(n) = 0$ and with equation (5) we have $v_{s-r}(n) = v_{s_n - r}(n)v_{s - s_n}(n + s_n - r) = 0$. Hence, again with Lemma 4.6, we have

$$\phi_s(n) = -\sum_{i=0}^{r-1} c_i(n + s - r)\phi_{s-r+i}(n) + v_{s-r}(n)\phi_r(n + s - r) = 0 + 0\phi_r(n + s - r) = 0.$$

So in this case the system corresponding to the ansatz of order s has a solution as well. Because of Lemma 4.5 we have a recurrence for $a + b$ of order $\operatorname{ord}(\phi_r) + r$. $\square$

Example 4.8. In Example 4.4 we have $r = 1$ and

$$\phi_1(n) = \begin{vmatrix} 1 & -(-1)^n \\ 1 & -1 \end{vmatrix} = -1 + (-1)^n,$$

for all $n \in \mathbb{N}$. The sequence ϕ_1 has order 2. Hence, $a + b$ has order at most $1 + \operatorname{ord}(\phi_1) = 3$. Indeed, we have seen a recurrence of order 3 in Example 4.4.

5 Further closure properties

C^2-finite sequences are not only closed under addition and multiplication. The following theorem gives several more closure properties.

Theorem 5.1. *Let $a, a_0, \ldots, a_{m-1}$ be C^2-finite. Then,*

1. *(shifts) $\sigma^k(a)$ is C^2-finite for every $k \in \mathbb{N}$,*

2. *(difference)* $\Delta(a) := \sigma(a) - a$ *is* C^2*-finite,*

3. *(partial sums)* $b = (\sum_{k=0}^{n} a(k))_{n\in\mathbb{N}}$ *is* C^2*-finite.*

4. *(subsequence)* $b = (a(nd))_{n\in\mathbb{N}}$ *is* C^2*-finite for every* $d \in \mathbb{N}$*,*

5. $b = (a(\lfloor n/m \rfloor))_{n\in\mathbb{N}}$ *is* C^2*-finite for every* $m \in \mathbb{N}$*,*

6. *(interlacing) if* $b = (b(n))_{n\in\mathbb{N}}$ *with* $b(n) = a_s(q)$ *such that* $n = qm + s$ *and* $0 \le s < m$*, then* b *is* C^2*-finite.*

Proof. (1), (2): Clear as the set of C^2-finite sequences is a difference ring by Theorem 3.3.

(3): We have $\sigma(b) - b = a$. Therefore, by Theorem 3.1, the sequence b is C^2-finite.

(4): Let $c_0 + c_1\sigma + \cdots + c_r\sigma^r \in \mathcal{R}_C[\sigma]$ be the annihilating operator of a. We have $\sigma^i b = (a(nd + id))_{n\in\mathbb{N}}$, for all $i \in \mathbb{N}$. For $c \in \mathcal{R}_C$ of order s we have

$$(c(nd + j))_{n\in\mathbb{N}} \in \langle (c(nd - s + j)_{n\in\mathbb{N}}, \dots, (c(nd - 1 + j)_{n\in\mathbb{N}}) \rangle_{\mathbb{K}},$$

for every $j \ge s$. Hence, by induction

$$(c(nd + j))_{n\in\mathbb{N}} \in \langle (c(nd)_{n\in\mathbb{N}}, \dots, (c(nd + s - 1)_{n\in\mathbb{N}}) \rangle_{\mathbb{K}},$$

for every $j \in \mathbb{N}$. In particular, the algebra

$$\mathbb{K}[(c(nd))_{n\in\mathbb{N}}, (c(nd + 1))_{n\in\mathbb{N}}, \dots]$$

is a Noetherian ring. Let $R \subsetneq \mathcal{R}_C$ be the ring containing the sequences $(c_i(nd + j))_{n\in\mathbb{N}}$, for all $i = 0, \dots, r$ and $j \in \mathbb{N}$. As the smallest ring containing finitely many Noetherian rings, this ring R is Noetherian. Let $i \in \mathbb{N}$. Using the recurrence for a and induction we have

$$\sigma^i b \in \langle (a(nd))_{n\in\mathbb{N}}, \dots, (a(nd + r - 1))_{n\in\mathbb{N}} \rangle_{Q(R)}.$$

Therefore, $\langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(R)}$ is finitely generated. Hence, by the proof of Theorem 3.1, b is C^2-finite.

(5): Suppose a satisfies $\sum_{i=0}^{r} c_i(n)a(n + i) = 0$, for all $n \in \mathbb{N}$. Then, we also have

$$\sum_{i=0}^{r} c_i(\lfloor n/m \rfloor) a(\lfloor (n + im)/m \rfloor) = \sum_{i=0}^{r} c_i(\lfloor n/m \rfloor) b(n + im) = 0,$$

for all $n \in \mathbb{N}$. Since $(c_i(\lfloor n/m \rfloor))_{n\in\mathbb{N}} \in \mathcal{R}_C$ for all $i = 0, \dots, r$ and $(c_r(\lfloor n/m \rfloor))_{n\in\mathbb{N}} \in \mathcal{R}_C^{\times}$, the sequence b is C^2-finite.

(6): For all $s = 0, \dots, m - 1$, the sequences $(a_s(\lfloor n/m \rfloor))_{n\in\mathbb{N}}$ are C^2-finite by part (5). Let

$$i_s(n) := \begin{cases} 1 & \text{if } n \equiv s \mod m, \\ 0 & \text{if } n \not\equiv s \mod m. \end{cases}$$

Then, $i_s \in \mathcal{R}_C$ for all $s = 0, \dots, m - 1$. Furthermore,

$$b(n) = \sum_{s=0}^{m-1} i_s(n) a_s(\lfloor n/m \rfloor), \quad \text{for all } n \in \mathbb{N}.$$

Since the set of C^2-finite sequences are a ring containing $\mathcal{R}_C$, the sequence b is C^2-finite. $\square$

Example 5.2. Let f denote the Fibonacci-sequence. With Corollary 3.4 and Theorem 5.1 the sequence

$$\left(\sum_{k=0}^{\lfloor 2n/3\rfloor} f((3k+1)^2)\right)_{n\in\mathbb{N}}$$

is C^2-finite.

6 Conclusion and outlook

Summarizing, we showed that C^2-finite sequences form a ring with respect to termwise addition and termwise multiplication. We derived several closure properties and methods to compute with C^2-finite sequences. Analogously to C^2-finite sequences one could define D^2-finite sequences as sequences which satisfy a linear recurrence with D-finite coefficients (with leading coefficient nonzero at every term). Theorem 3.1 and Theorem 3.3 can be adapted straightforward to show that these D^2-finite sequences form a ring. However, it is not known whether the Skolem-Mahler-Lech Theorem holds for D-finite sequences [2]. Hence, some of the ideas for computing in the C^2-finite sequence ring cannot be carried over immediately to D^2-finite sequences.

For D-finite functions, i.e., formal power series satisfying a linear differential equation with polynomial coefficients, an analogous construction has been carried out [9]: D^2-finite functions satisfying a linear differential equation with D-finite function coefficients. An advantage of this setting is that D-finite functions form an integral domain and one does not have to deal with zero divisors. D^2-finite functions satisfy most closure properties known for D-finite functions (except for the Hadamard product). From this it can be derived that the construction can be iterated to build D^k-finite functions.

Similarly, one can define C^{k+1}-finite sequences as sequences satisfying a linear recurrence with C^k-finite sequence coefficients. The proof of Theorem 3.3 can be adapted by iterating the construction of the Noetherian ring R. This shows that C^k (and D^k-finite) sequences are difference rings as well.

A useful feature of D-finite sequences is that their generating functions are D-finite as well and that the defining difference and differential equations can be computed from one another. This is often exploited in proofs or simplification of identities. Also most of the results of Theorem 5.1 would typically be proven by switching between those two representations.

Since D^2-finite functions are not closed under Hadamard product, there cannot be a one-to-one correspondence to D^2-finite sequences. Still, it seems worthwhile to investigate the relationship between these sets. First ideas on the nature of generating functions of C^2-finite sequences have been presented in [16]. It would be interesting to explore this further and to derive computational properties.

References

[1] Michael F. Atiyah and Ian G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley-Longman, 1969.

[2] Jason P. Bell, Stanley N. Burris, and Karen Yeats. On the set of zero coefficients of a function satisfying a linear differential equation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 153(2):235–247, 2012.

[3] Jean Berstel and Maurice Mignotte. Deux propriétés décidables des suites récurrentes linéaires. *Bulletin de la Société Mathématique de France*, 104:175–184, 1976.

[4] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov, Release 1.1.0 of 2020-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[5] Graham Everest, Alf van der Poorten, Igor Shparlinski, and Thomas Ward. *Recurrence Sequences.* Mathematical Surveys and Monographs. American Mathematical Society, 2015.

[6] Stefan Gerhold. *Combinatorial Sequences: Non-Holonomicity and Inequalities.* PhD thesis, Johannes Kepler University Linz, 2005.

[7] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2021. http://www.oeis.org.

[8] Antonio Jiménez-Pastor and Veronika Pillwein. Algorithmic Arithmetics with DD-Finite Functions. In *Proceedings of ISSAC 2018, New York, NY, USA, July 16–19, 2018*, pages 231–237, 2018.

[9] Antonio Jiménez-Pastor and Veronika Pillwein. A Computable Extension for Holonomic Functions: DD-Finite Functions. *J. Symb. Comput.*, 94:90–104, 2019.

[10] Manuel Kauers. The Holonomic Toolkit. In *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts and Monographs in Symbolic Computation, pages 119–144. Springer, 2013.

[11] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron.* Texts and Monographs in Symbolic Computation. Springer, 2011.

[12] Emrah Kilic, Ilker Akkus, and Hideyuki Ohtsuka. Some generalized fibonomial sums related with the gaussian q-binomial sums. *Bulletin mathématiques de la Société des sciences mathématiques de Roumanie*, 55, 2012.

[13] Tomer Kotek and Johann A. Makowsky. Recurrence relations for graph polynomials on bi-iterative families of graphs. *Eur. J. Comb.*, 41:47–67, 2014.

[14] Joël Ouaknine and James Worrell. Decision Problems for Linear Recurrence Sequences. In *Lecture Notes in Computer Science*, pages 21–28. Springer, 2012.

[15] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger. $A = B$. CRC Press, 1996.

[16] Thotsaporn Aek Thanatipanonda and Yi Zhang. Sequences: Polynomial, C-finite, Holonomic, ..., 2020. https://arxiv.org/pdf/2004.01370.

[17] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*, 2021. https://www.sagemath.org.

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